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ADAPTIVE CONTROL TECHNIQUES FOR LARGE SPACE STRUCTURES

Annual Technical Report

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This report summarizes the research performed during the period 1 June 1985 to 31 May 1986. on adaptive control techniques for Large Space Structures (LSS). The research effort concentrated on two areas: (1) on-line robust design from identified models – what is referred to here as adaptive calibration; and (2) an analysis of slow-adaptation for adaptive control of LSS. The report summarizes the results obtained in these areas and also includes Appendices which contain technical articles.

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INTRODUCTION

The Large Space Structure (LSS) research program was originally formulated in late 1982 in response to the increasing concern that performance robustness of Air Force LSS type systems would be inadequate to meet mission objectives. In particular, uncertainties in both system dynamics and disturbance spectra characterizations (both time varying and stochastic uncertainty) significantly limit the performance attainable with fixed gain, fixed architecture controls. Therefore, the use of an adaptive system, where disturbances and/or plant models are identified prior to or during control, gives systems designers more options for minimizing the risk in achieving performance objectives.

The aim of adaptive control is to implement in real time and on line as many as possible of the design functions now performed off line by the control engineer to give the controller "intelligence." To realize this aim, both a theory of stability and performance of such inherently nonlinear controls is essential as well as a technology capable of achieving the implementation.

The issues of performance sensitivity, robustness, and achievement of very high performance in an LSS system can be effectively addressed using adaptive algorithms. The need to identify modal frequencies, for example, in high-performance disturbance rejection systems has been shown in ACOSS (1981) and VCOSS (1982). The deployment of high-performance optical or RF systems may require on-line identification of critical modal parameters before full control authority can be exercised. Parameter sensitivity, manifested by performance degradation or loss of stability (poor robustness) may be effectively reduced by adaptive feedback mechanizations. Reducing the effects of on-board disturbance rejection) is particularly important for planned Air Force missions. For these cases, adaptive control mechanizations are needed to produce the three-to-five orders-of-magnitude reductions in line-of-sight jitter required by the mission.

Research is essential to identify the performance limitations of adaptive strategies for LSS control both from theoretical and hardware mechanization viewpoints. The long range goal of this proposed research program is to establish guidelines for selecting the appropriate strategy, to evaluate performance improvements over fixed-gain mechanizations, and to examine the architecture necessary to produce a practical hardware realization. The initial thrust, however, is to continue to build a strong theoretical foundation without losing sight of the practical implementation issues.

RESEARCH OBJECTIVES

The aims of this research study are to extend and develop adaptive control theory and its application to LSS in several directions. These include:

(1) **Theoretical Development:** The initial emphasis has been on slow adaptation, since this covers many LSS situations. Later on we will examine fast adaptation. The theory developed here will provide for:

    (a) estimates of robustness, i.e., stability margins vs. performance bounds;

    (b) estimates of regions of attraction and rates of parameter convergence to these regions;

    (c) extension of the present linear finite dimensional adaptive theory to include nonlinear and infinite dimensional plants and controller structures; and

    (d) extensions to decentralized systems.

(2) **Parameter Adaptive Algorithms:** Assesses the behavior of different algorithms, including: gradient, recursive least squares, normalized least mean squares, and nonlinear observer (e.g., Extended Kalman Filter).

(3) **Parametric Models:** Assess the impact of model choices. In particular we will examine the effect of explicit and implicit model choices. An explicit model, for example, is a transfer function whose coefficients are all unknown. In an implicit model model transfer function, the coefficients would be functions of some other parameters. Implicit models usually arise from physical or experimental data, whereas explicit models are selected for analytical convenience.

(4) **Adaptive Nonlinear Control:** Although our early effort is to study adaptive linear control, there are many LSS situations where the control is nonlinear, e.g., large angle maneuvers, slewing.

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CURRENT STATUS

At the present time we stand at the beginning stages of the theoretical development in adaptive control. The result of recent efforts are contained in the selected papers in the Appendix and the references therein. A summary of earlier efforts is contained in the recently published textbook *Stability of Adaptive Systems: Passivity and Averaging Analysis*, MIT Press, 1986. This publication is an outgrowth of research supported under this contract and involved a considerable amount of collaborative effort among several researchers in the field of adaptive control. The text discusses adaptive systems from the viewpoint of stability theory. The emphasis is on methodology and basic concepts, rather than on details of adaptive algorithm. The analysis reveals common properties including causes and mechanisms for instability and the means to counteract them. Conditions for stability are presented under slow adaptation, where the method of averaging is utilized. In this latter case the stability result is *local*, i.e., the initial parametrization and input spectrum is constrained. Based on this analysis, a conceptual framework is now available to pursue the issues of slow adaptive control of LSS.

To remove the restrictiveness of slow adaptation requires an understanding of the transient behavior of adaptive systems. A preliminary investigation is reported in Kosut et al. (1986) which is reprinted in the Appendix. The transient behavior of not-slow or even rapid adaptation is a significant problem in the adaptive control of LSS, e.g., rapid retargeting.

Another approach to adaptive control is to calibrate (or tune) the controller based on a current estimate of the LSS model. This involves not just knowing *one* model, but rather, a model *set*. This problem, which we refer to as *adaptive calibration*, is essentially that of developing a technique of on-line robust control design from an identified model. Although we have worked on this problem for some time it is only recently that we have established a theoretical basis for estimating model error from system identification [see Kosut (1986), a reprint is in the Appendix]. This research has raised many new questions which need to be considered, e.g., what is the appropriate robust controller parametrization; how does it relate to model parametrization; how to iterate on the data if the estimate of model error is too large; what are the heuristics for experiment design.

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R.L. Kosut, "Towards an on-line procedure for automated robust control design: the adaptive calibration problem," presented at *1986 ACC*, June 1986.

R.L. Kosut,[#] I.M.Y. Mareels, B.D.O. Anderson, R.R. Bitmead, and C.R. Johnson, Jr., "Transient Analysis of Adaptive Control", submitted to *IFAC, 10th World Congress,* Munich, Germany, July 1987.

R.L. Kosut,[#] and R.R. Bitmead, "Fixed-Point Theorems for Stability Analysis of Adaptive Systems", *Proc. IFAC Workshop on Adaptive Systems,* Lund, Sweden, July, 1986.

R.L. Kosut,[#] and R.R. Bitmead, "Linearization of Adaptive Systems: A Fixed-Point Analysis", submitted, *IEEE Trans. on Circuits and Systems; Special Issue on Adaptive Systems,* Sept. 1987.

I.M.Y. Mareels, R.R. Bitmead, M. Gevers, C.R. Johnson, Jr., R.L. Kosut,[#] and M.A. Poubelle, "How Exciting Can a Signal Really Be?" to appear, *Systems and Control Letters.*

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[#] Research performed while R.L. Kosut was a Visiting Fellow at the Australian National University.

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R.L. Kosut and M.G. Lyons, "Issues in Control Design for Large Space Structures", *Adaptive Systems: Theory and Applications,* Editor: K.S. Narendra, Plenum Press, 1986.

In Preparation

R.L. Kosut, "Identification and On-Line Robust Control of Large Space Structures", *Sixth VPI AU/AIAA Symp. on Large Structures,* Blacksburg, VA, June, 1987.

R.L. Kosut, "Adaptive Control of Large Space Structures", *Large Space Structures: Dynamics and Control,* edited by S.N. Atluri and A.K. Amos, Springer-Verlag, New York.

R.L. Kosut, "Identification of Flexible Structures for Control Design", *1987 ASME Vibration Conference,* Boston, MA, Sept. 1987.

COLLABORATIVE RESEARCH EFFORT

It should be emphasized, and acknowledged, that a great deal of collaborative effort has been, and is being, expended by several researchers in the field of adaptive control. The text referred to before is in part due to the two visits by Dr. Kosut to the Department of System Engineering at the Australian National University. Support for these visits has come from this contract, a travel grant from the NSF International Program (INT-85-13400), and a Visiting Fellow Award from the Australian National University.

FUTURE DIRECTIONS

Based on our recent results as reported here, we envision near-term activity in several directions, including:

(a)   transient analysis of adaptive control;

(b)   analysis of adaptive calibration;

(c)   decentralized control structures;

(d)   effect of nonlinear and infinite dimensional phenomena;

(e)   effect of different algorithms and parametrizations.

APPENDIX

Issues in Control Design for Large Space Structures

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Abstract

The development of a design methodology for the control of Large Space Structures (LSS) involves many different issues. In this paper we present a selective discussion of the theoretical and practical issues that seem most relevant. The discussions cover various types of control design procedures, including both robust (non-adaptive) as well as adaptive, with an emphasis on their practical use.

1. LSS Control Problem Setting

1.1 Control Design Objectives

Problems associated with vibration control and accurate pointing of LSS systems typically involve a combination of the following control-performance objectives.

1. modal damping augmentation to enhance transient settling or improve quasi-static vibration propagation behavior,

2. stabilization of the attitude control system,

3. eigenvector modification to reject narrow band steady-state disturbances, and

4. maneuver load management to minimize structural loads or modal excitation (transient or steady-state).

1.2 Modeling

The basis for selecting a control strategy must include an adequate description of the relevant structural dynamics together with a description of how system performance is to be measured. Initially, continuum models were suggested as the basis for proper system design since discretization of the model could be postponed or eliminated. Unfortunately, practical spacecraft configurations do not present simple boundary conditions or simple shapes, hence partial differential equation (p.d.e.) representations are nearly impossible to write. However such continuum models have provided useful insight into appropriate discrete representations. Finite element models can provide adequate fidelity, at least over the frequency range needed for the control design model, and are supported with sophisticated software tools easily adapted to the needs of control design [1].

1.3 Two-Level Control Architecture

The natural structural properties of LSS systems compel the use of a two-level control system architecture as shown in Figure 1. The two levels are a colocated rate-damping control system and a noncolated high performance control system. The colocated system consists typically of rate damping devices, either active or passive, and requires a coarse knowledge of system dynamics. These are inherently robust but yield low performance. They essentially provide a wide-band, Low-Authority Control (LAC) and are often referred to as the LAC-system.
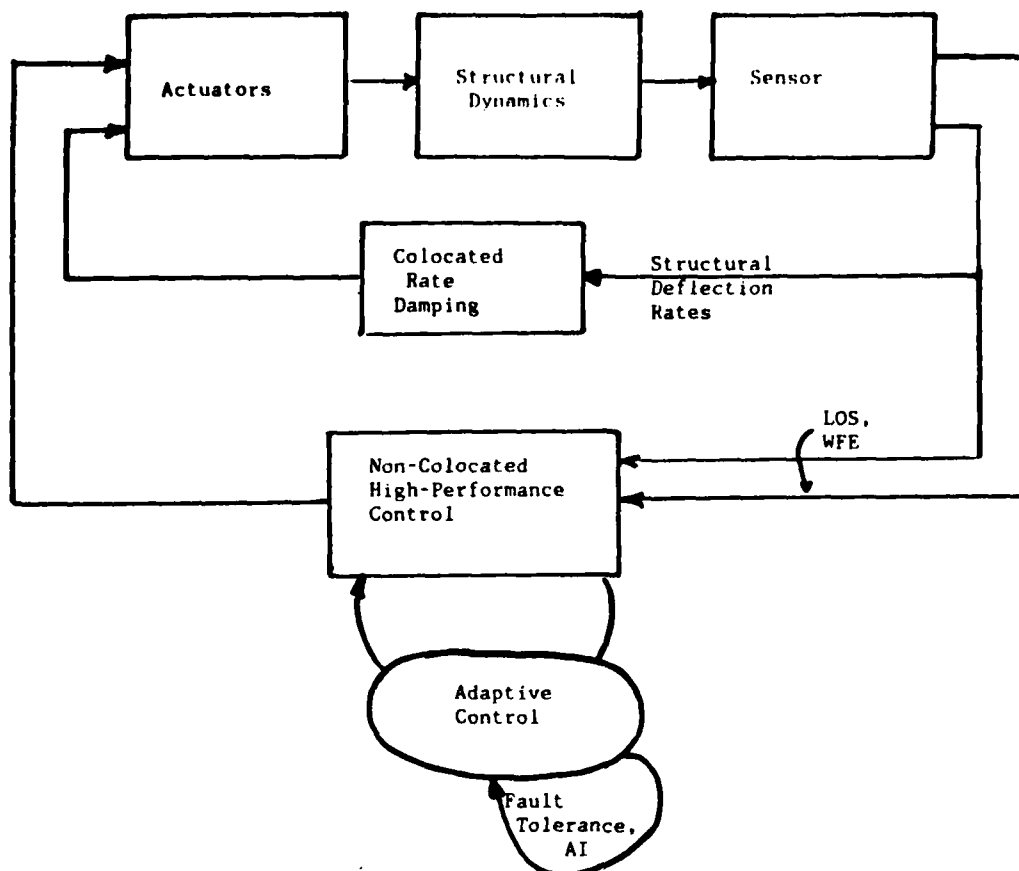
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Figure 1: Two Level Control Architecture for LSS Systems.

The high performance control is non-colocated and requires accurate knowledge of critical modes, and hence, is very sensitive to disturbance and structural parameter variations. This controller system is essentially a narrow-band, High-Authority Control (HAC), and is referred to as the HAC-system. Typically, HAC provides high damping and mode shape adjustment in selected modes to meet performance requirements.

LAC synthesis principally involves passivity methods and rate feedback mechanizations, usually with colocated actuators and sensors [2].

HAC synthesis, in addressing performance goals associated with dynamic wavefront and line-of-sight error suppression, requires high modal damping and mode shape changes. Hence, HAC is dependent on accurate narrow-band models. For such requirement, it is essential that control design techniques manage both dependence on model fidelity and system gain in regions where model fidelity is poor. This has generally been accomplished using fixed-gain robust control theory, [4]. With this architecture it is likely that only the HAC would be tuned by an adaptive system since the LAC is inherently robust.

1.4 Adaptive Techniques

In general, uncertaintites in both disturbance spectra and system dynamical characteristics limit the performance obtainable with fixed gain, fixed order control, e.g. HAC system. The use of an adaptive control mechanization where disturbance and/or plant dynamics are identified prior to or during control, gives system designers more options for minimizing the risk in achieving performance benchmarks.

In the case of LSS systems, the performance levels are extremely high. Hence it is necessary that disturbance and plant models are accurately known. Since model data obtained from ground testing is unlikely to sufficiently match the actual on-orbit system, it follows
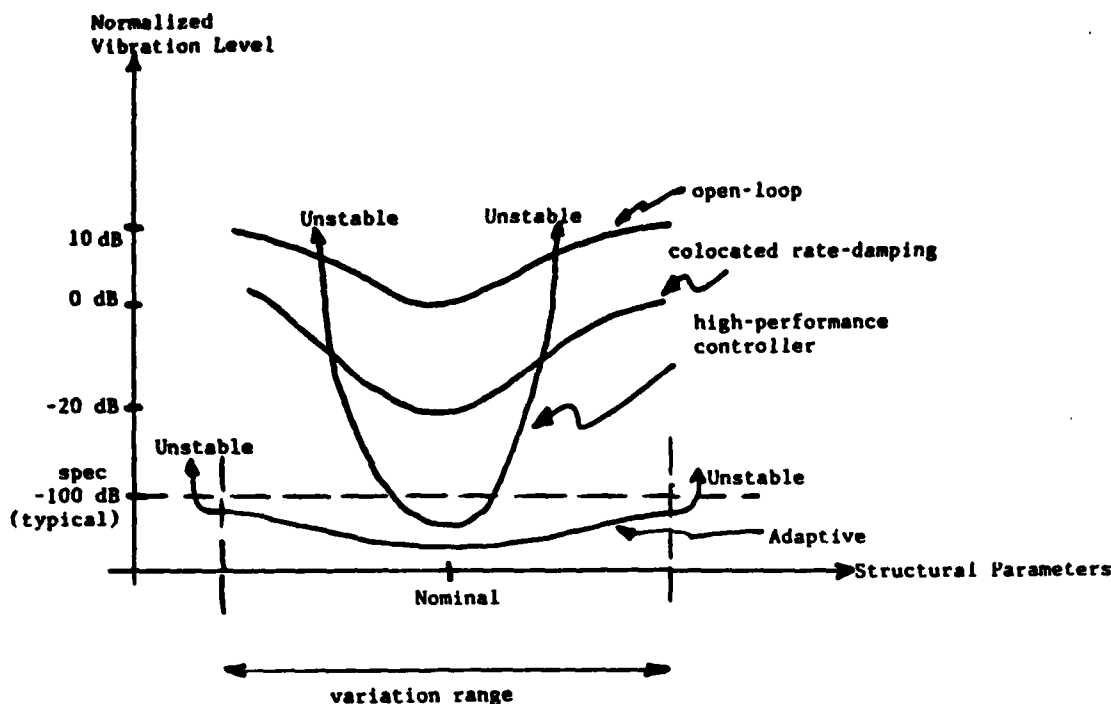
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Figure 2: Closed-loop System Performance vs. Structural Parameter Variations.

that on-line procedures are needed for identification and control.

The generic properties of closed-loop system performance vs. structural parameter variations are depicted in Figure 2.

2. Control Design for HAC/LAC Architecture

In this section we wil discuss the steps involved in control design for the HAC/LAC architecture. Although the architecture is specialized, the control design methodology is not and can be quite general. We will discuss three methodologies for design: (1) an LQG based methodology whose genesis is the ACOSS/VCOSS programs, and (2) a more recent approach involving what is known as "Q-parametrization" and H_∞-optimization". These latter methods are frequency domain oriented rather than state-space oriented like the LQG approach. (3) We will also discuss an adaptive control strategy which can be utilized for online self-tuning. We refer to this approach as "adaptive calibration".

2.1 Limitations of Design

Independent of the design method, the defining characteristic of the vibration control problem is that there are an infinite number (theoretically) of elastic modes, with low natural damping, and the controller bandwidth extends over a significant number of these modes (Figure 3). The low frequency modes interact not only with the attitude controller but contribute directly to the deformation geometry of the structure which itself may require accurate control. Proper control synthesis requires that performance criteria be precisely formulated or the control problem is ill-posed.

The control design approach must properly handle the poorly known higher frequency modes by not destabilizing them while controlling the low frequency modes. Indeed, no matter where the controller roll-off frequency is situated, the infinite nature of the modal

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CONTROLLER BANDWIDTH

SYSTEM MODES

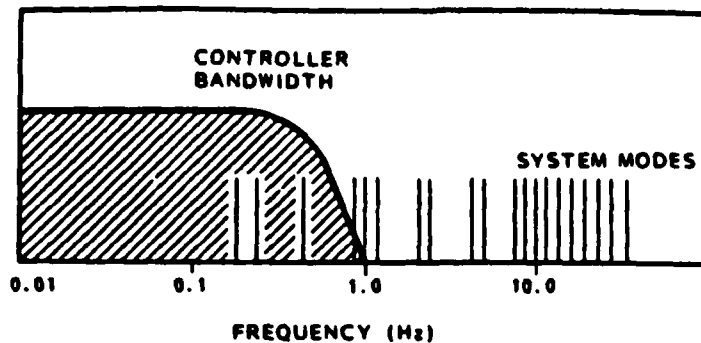0.01   0.1   1.0   10.0

FREQUENCY (Hz)

Figure 3: Flexible Structure Mode Location and Controller Bandwidth.

spectrum implies that there will be modes within and beyond the roll-off region. Furthermore, destabilization is likely and almost certain to occur in the roll-off region, a situation which can only worsen for closely packed modes and low natural damping. This phenomenon sometimes referred to as "spillover" is one of the most crucial problems faced by the control designer. In more general terms, spillover can be viewed as an aspect of the problem of robust control design; this will be discussed more in a later section.

2.2 Modeling of Flexible Spacecraft

A central issue in the active control of space structures is the development of "correct" mathematical models for the open and closed loop dynamical plants. Programs such as NASTRAN and SPAR are the primary current tools for generating dynamical models of conceptual spacecraft whose structure cannot be idealized by simple models of beams, plates, and beams with lumped masses.

Finite element structural programs generally provide the control designers with a set of modal frequencies and a set of mode shapes (eigenvectors) corresponding to appropriate boundary values (e.g. free-free modes). These eigenvectors are given in discretized form, i.e. a set of modal displacements in the x, y and z directions at each nodal station. In some cases, modal rotations are also required. In addition, coordinates and a "map" of the structure's nodes must be provided to allow the reconstruction of physical displacements in terms of their modal expansions.

The important point here is that, for any nontrivial flexible satellite configuration, the volume of information is so large that the data handling must remain entirely within the computer and its mass-storage facilities. Development of this database, in a form usable by control synthesis software, is a fundamental necessity for the synthesis and evaluation of complex control which require modal truncation, actuator/sensor location and type changes, and evaluation of system performance for parameter and system order changes. Preparation of a structure for controls is a major part of the overall effort required to develop structural control systems.

2.3 Nonlinear Models

For single-body monolithic structures, the fine-pointing attitude dynamics are subsumed in the rotational rigid body modes included in the modal matrix. When only "small" motions of a space structure are being considered, the conventional linear structural dynamics analyses (NASTRAN and SPAR) are adequate, and the rigid-body modes are formally handled together with the elastic modes, even though the actuators necessary to control them will be different, in general, from those used to control elastic vibrations. When larger attitude angles need to be considered, if the angular rates remain small, the linear equations are still applicable provided that the rigid-body modes are now given in terms of three attitude angles which then constitute the first three modal coordinates. The displacements are then
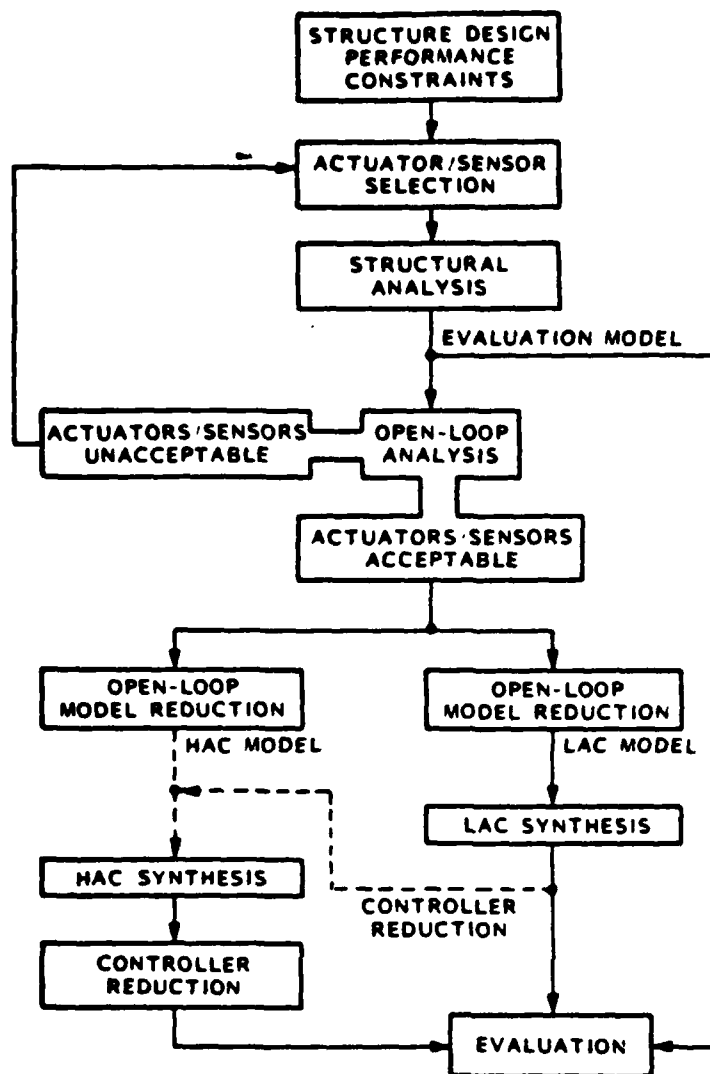
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Figure 4: Analytical Control-Design Procedure.

interpreted as the linear deformations of the structure with respect to the rotated frame. This procedure removes the kinematic nonlinearities resulting from the linear stretching of the structure under the classical rigid-body modes. However, for large angular rates, nonlinear dynamic effects have to be modeled, even though structural deformations can still be represented by linear equations.

2.4 Two-Level Control Design: The HAC/LAC Methodology

The two-level approach consists of a wide-band, low-authority control (LAC) and a narrow-band, high-authority control (HAC). HAC provides high damping or mode-shape adjustment in a selected number of modes to meet performance requirements. LAC, on the other hand, introduces low damping in a wide range of modes for maximum robustness. Figure 4 shows the control design procedure with integrated LAC and HAC designs.

LAC is usually implemented with colocated sensors and actuators. However, the theory, based on the work of Aubrun, is applicable to multiple actuators/sensors with cross-feedback and possible filters [2].

HAC uses a collection of sensors and actuators not necessarily colocated. Selecting the increase in damping ratio is realized by any number of methods including LQG with frequency shaping, Q-parametrization, or H_∞-optimization. These methods provide roll-off over desired frequency regions. HAC may destabilize modes not used in the design. LAC is,
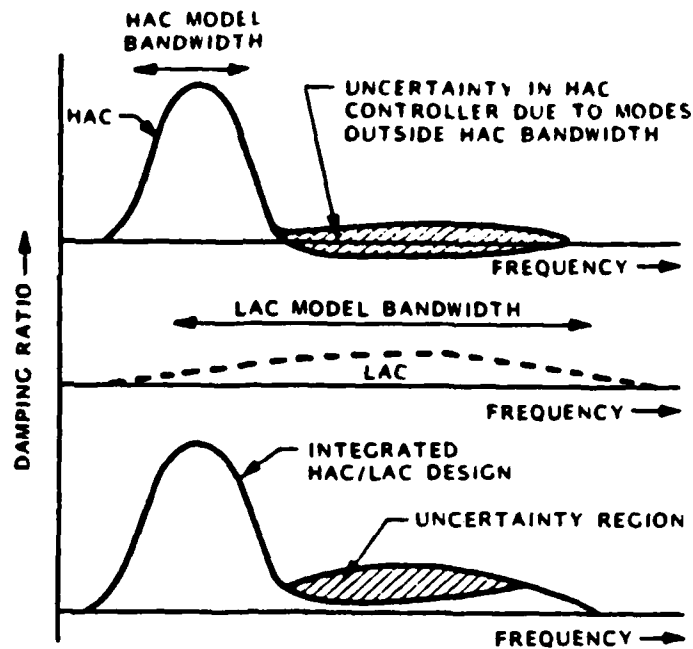
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Figure 5: Need to Integrate HAC and LAC.

therefore, necessary to "clean up" problems created by HAC.

The need to integrate HAC with LAC is shown in Figure 5. HAC is based on models valid over a limited frequency region. It produces large increases in damping ratio and disturbance rejection in the frequency range of interest. The effect of the HAC controller on modes not used in the control design and outside the controller bandwidth may be stabilizing or destabilizing. LAC is designed to provide protection such that adequate damping is provided in the mode most adversely perturbed by HAC. With reference to Figure 5, the LAC moves the entire uncertainty region above the zero level damping ratio.

In the next few sections, a more in-depth discussion of the blocks in Figure 4 will be presented, in particular, actuator/sensor location, model and controller reduction methods, and HAC/LAC synthesis. These methodologies rely on certain properties of feedback control: this raises the issue of robust control design which is fundamental to the whole design philosophy of feedback, especially for LSS, and this will be discussed first.

2.5   Robust Control Design

This section will describe how to evaluate the robustness of a control design. The evaluation is independent of the methodology used to achieve a particular design. To illustrate the technique we will consider the robust control problem of vibration suppression with unmodeled high frequency dynamics. Figure 6 shows the control system where $P(s)$ is the plant transfer function matrix from actuator inputs to LOS sensor measurements, and where $C(s)$ is the controller transfer function matrix. Neglecting the rigid body modes in $P(s)$ and assuming infinite bandwidth sensors and actuators,

$$P(s) = \sum_{k=1}^{\infty} G_k(s)$$

where

$$G_k(s) = \frac{1}{s^2 + 2\xi_k \omega_k s + \omega_k^2} M_k.$$

Suppose that n of the modes are known. Let $P_n(s)$ denote the known part of $P(s)$.
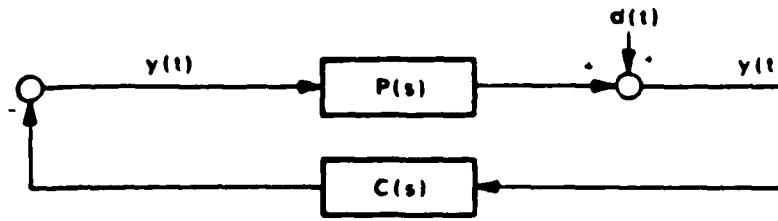
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Figure 6: Vibration Suppression Control System.

For example, $P_n(s)$ can be obtained from $P(s)$ by modal truncation, i.e., the first n-modes of $P(s)$ are retained. One can ask the question: is this the best choice for a given model order n? In general, it depends on what is meant by "best". For closed loop control, it is usually better to retain those n-modes which most affect the closed-loop performance. How to select these modes will be discussed in the section on model reduction.

Assuming the modes have been selected, define model error as

$$\delta(s) = P(s) - P_n(s) = \sum_{k \in \Omega_n} G_k(s).$$

Observe that $\delta(s)$ is stable because both $P(s)$ and $P_n(s)$ are stable. Hence, it can be shown that the closed loop system is stable if

$$\bar{\sigma}\left[\delta(j\omega)\right] < \delta_{sm}(j\omega) = 1/\bar{\sigma}\left[Q_n(j\omega)\right]$$

where $Q_n(s)$ is given by $Q_n(s) = C(s)[I + P_n(s)C(s)]^{-1}$ and $\bar{\sigma}(\cdot)$ denotes the maximum singular value of the matrix argument. The quantity $\delta_{sm}(\omega)$ is referred to as the "stability margin", hence, the subscripts "sm". (See [3,4].)

The stability robustness test depends on the location of uncertainty. Additive perturbations such as those just discussed result in the test as shown. The table in Figure 7 shows a variety of stability margins corresponding to generic forms of model error. In Figure 7, P = plant, C = control, M nominal model, and δ = model error. The stability margin is expressed as a function of C and M which are known quantities. Examples of some model error testts are shown in Figure 8 for the CSDL #2 VCOSS model.

2.6 Performance Robustness

The stability robustness tests can be extended to evaluate performance robustness to model error. The evaluation is determined by how performance is measured. Consider the closed loop system

$$y(t) = H(s)d(t)$$

where $H(s)$ is the closed loop transfer function. Although $d(t)$ is not precisely known, it can be considered as the output of a weighting filter $W(s)$ driven by "noise" $w(t)$ so that $d(t) = W(s)w(t)$.

Typical performance bounds depend directly on the frequency dependent quantity $[H(j\omega)W(j\omega)]$. A natural frequency domain performance criterion is then

$$\bar{\sigma}[H(j\omega)W(j\omega)] \leq \rho(\omega)$$

where $\rho(\omega)$ is selected on the basis of power, energy, and magnitude specifications on the output signals. In terms of model error, performance specification is satisfied if

$$\bar{\sigma}[\delta(j\omega)] \leq \delta_{pm}(\omega)$$

where $\delta_{pm}(\omega)$ is the performance margin given by

$$\delta_{pm}(\omega) = [1 - \rho_n(\omega)/\rho(\omega)]\delta_{sm}(\omega)$$

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GENERIC FORM OF MODEL ERROR	SOURCE OF MODEL ERROR IN SPACECRAFT SYSTEM	STABILITY MARGIN — GUARANTEED STABILITY IF $\bar{\sigma}(\Delta) < \delta_{SM}$	CONTROLLED SPACE
P = ACTUAL PLANT M = MODEL Δ = MODEL ERROR			
ADDITIVE $P = M + \Delta$	• NEGLECTED RESIDUAL MODES, e.g., – SPILLOVER – REDUCED ORDER MODELING – UNCERTAIN INTERACTING STRUCTURAL MODES	$\delta_{SM} = 1/\bar{\sigma} \, [C(I + MC)^{-1}]$	
OUTPUT MULTIPLICATIVE $P = (I + \Delta)M$	• SENSOR ERRORS – MISALIGNMENTS – BANDWIDTH – SCALE FACTORS • NEGLECTED HIGH FREQUENCY PHENOMENA, e.g., – MODEL APPROXIMATIONS	$\delta_{SM} = 1/\bar{\sigma} \, [(I + MC)^{-1} MC]$	
INPUT MULTIPLICATIVE $P = M(I + \Delta)$	• ACTUATOR ERRORS – BANDWIDTH – NONLINEARITIES – PARASITICS – QUANTIZATION – FRICTION – STICTION	$\delta_{SM} = 1/\bar{\sigma} \, [(I + CM)^{-1} CM]$	

Figure 7: Source of Model Error in Spacecraft System.

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FEEDBACK SYSTEM

ADDITIVE TEST

$P = M + \Delta$

$\|\Delta\| < \|C(I + MC)^{-1}\|^{-1}$

INPUT MULTIPLICATIVE TEST

$P = M(I + \Delta)$

$\|\Delta\| < \|(I + CM)^{-1} CM\|^{-1}$

OUTPUT MULTIPLICATIVE TEST

$P = (I + \Delta) M$

$\|\Delta\| < \|(I + MC)^{-1} MC\|^{-1}$

Figure 8: Results of Model Error Tests.

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and $\rho_n(\omega)$ is the performance of the nominal closed loop system $H_n(s)$ with no model error. Then,
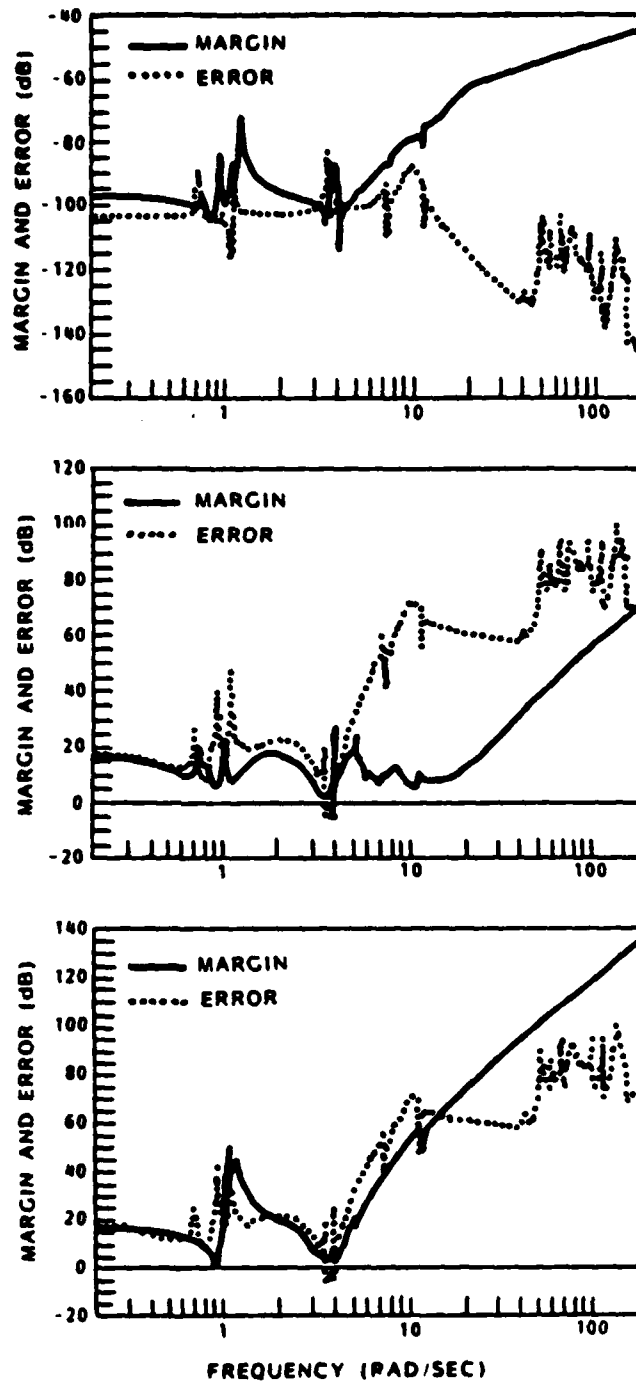
$$\rho_n(\omega) = \bar{\sigma}[H_n(j\omega)W(j\omega)]$$

which must always be smaller than $\rho(\omega)$ in order for $\delta_{pm}(\omega)$ to be meaningful. Note that $\delta_{pm}(\omega) > \delta_{sm}(\omega)$ as would be expected since performance includes stability. As before, the location of uncertainty modifies the calculation of $\delta_{pm}(\omega)$.

2.7 Usefulness of Stability/Performance Robustness Tests

The stability/performance robustness tests are indispensible in obtaining a realistic preliminary design. They are used in a number of places in the design cycle to establish the HAC/LAC gains, effect of actuator/sensor dynamics, and the criteria for model and controller reduction, which will be discussed in the next section. The tests are also invaluable in establishing criteria for online system identification and control, which will be discussed later on in this section.

2.8 Model Reduction

In general, the requirements for model reduction for active control of large space structures must include the following:

1. The reduced model should be suitable for control design and synthesis. It should incorporate all features critical for the selection of a feedback structure and control gains.

2. The reduced model should accurately incorporate actuator effectiveness, sensor measurements and disturbance distribution [1].

3. The dynamical characteristics of interest in the structure should be represented in the reduced model.

A basic methodology for model reduction which has been used successfully in ACOSS/VCOSS and a number of other programs such as internal balancing, is now described. Other approaches also exist which will be discussed in the sequel.

2.9 Internal Balancing

To determine the most important modes for control design, many criteria must be considered including controllability, disturbability, observability in performance, and observability in the measurements. Any mode which is highly controllable, observable, and disturbable must clearly be included in the design model: however highly controllable-but-unobservable modes, for example, are difficult to judge. Moore [5] has developed an "internal balancing" approach whereby asymptotically stable linear models are transformed to an essentially unique coordinate representation for which controllability and observability rankings are identical. The definition of internally balanced coordinates follows:

Definition: An asymptotically stable model

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx\end{aligned}$$

is *internally balanced* over $[0, \infty]$ iff

$$\int_0^\infty e^{At} BB^T e^{A^T t} dt = \int_0^\infty e^{A^T t} C^T C e^{At} dt = \sum{}^2$$

where

$$\sum{}^2 = diag[\sigma_1^2\ \sigma_2^2\ \ldots \sigma_n^2] \quad i \geq j, \ \sigma_i^2 \geq \sigma_j^2.$$
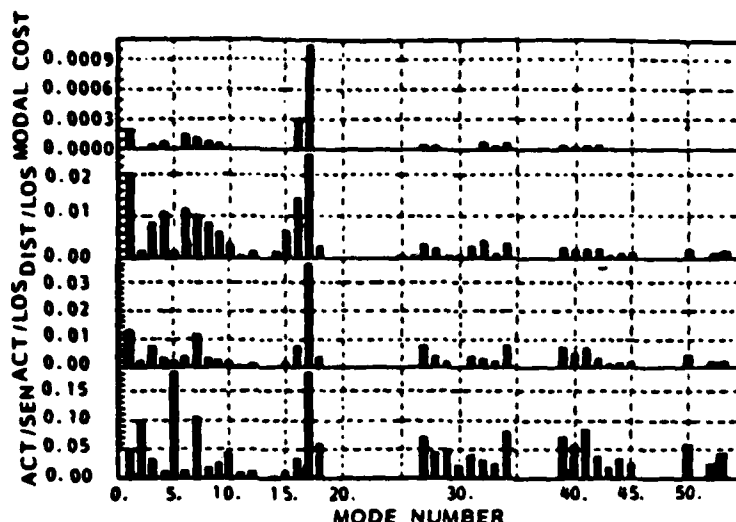
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Figure 9: Open Loop Modal Analysis.

Notice that the balanced representation is such that the controllability Gramian and observability Gramian are equal and diagonal. The σ_i's are termed "second-order modes." In general, the required transformation "scrambles" the original coordinate system such that the physical meaning of the states is lost.

However, for lightly damped structural models with decoupled dynamics, the internally balanced coordinate representation is approximately equal to a scaled representation of the model states. Thus it is possible to write approximate formulae for the states in terms of the original model. Three modal rankings are considered:

- disturbance inputs to LOS

- actuator inputs to LOS

- actuator inputs to sensor outputs

These "second-order modes" rankings give important evaluations about which modes to retain and validity of a actuator/sensor placement. These rankings are shown in Figure 9 along with LOS modal cost [6] computed using the colored noise disturbance.

Here the absolute values of the modal costs (for the VCOSS 1 model) are used. The RMS second-order modes and modal costs are plotted versus mode number in Figure 9. Immediately evident is the clustering of these modal phenomena. The disturbance effect as seen through the line-of-sight is constrained to clusters of modes as is the ability to measure and control the model. The coincidence of the controllable clusters and disturbable clusters indicates a favorable actuator/sensor configuration for the problem.

2.10 Frequency Weighted Balanced Realizations

Balanced realization model reduction can be extended to finding a reduced model $P_n(s)$ of a high order model $P(s)$ such that

$$\sup_{\omega} \bar{\sigma} \left\{ W_o(j\omega)[P(j\omega) - P_n(j\omega)]W_i(j\omega) \right\} < 1$$

where $W_o(s)$ and $W_i(s)$ are output and input frequency dependent weighting matrices. These can be chosen to reflect closed-loop requirements on model error, vis a vis, frequency domain stability and performance margins. For example, stability of the closed loop system with $C(s)$ designed from $P_n(s)$ is guaranteed if

$$\begin{aligned}
W_o(s) &= I \\
W_i(s) &= C_n(s)[I + P_n(s)C_n(s)]^{-1}.
\end{aligned}$$

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The problem is that $W_i(s)$ is dependent on $P_n(s)$ which is unknown. The let out is that its shape is partially determined by the performance specifications, thus, we can make an initial guess. This technique is referred to as "advanced loop shaping." This involves an iterative problem which is solvable via successive approximation.

2.11 Compensator Order-Reduction

An alternative to plant order reduction is to design a high order compensator and then reduce the compensator order. Let $C(s)$ denote a high order compensator of order N designed to control $P(s)$ of order N or larger. Let $C_n(s)$ denote a reduced version of $C(s)$ of order $n < N$. Motivated by the stability robustness theory, view $C(s) - C_n(s)$ as a perturbation. Hence, the closed loop system with $P(s)$ and $C_n(s)$ is stable if

$$\sup_\omega \overline{\sigma} \left\{ W(j\omega)[C(j\omega) - C_n(j\omega)] \right\} < 1$$

where $W(s) = (I + P(s)C(s))^{-1}P(s)$.

The weight $W(s)$ is stable because the high order control $C(s)$ stabilizes the closed loop system. In this case $W(s)$ is known and we can apply internal balancing to find $C_n(s)$. The disadvantage to this method is that it is necessary to find a high-order compensator. The advantage is that once it is found, internal balancing applies immediately since the weights are known. On the other hand, direct plant order reduction does not involve control design for the high order plant, but does involve an iterative process since the weights are functions of the (unknown) reduced model.

2.12 Low-Authority Control Design

LAC systems, when applied to structures, are vibration control systems consisting of distributed sensors and actuators with limited damping authority. The control system is allowed to modify only moderately the natural modes and frequencies of the structure. This basic assumption, combined with Jacobi's root perturbation formula, leads to a fundamental LAC formula for predicting algebraically the root shifts produced by introducing a LAC structural control system. Specifically, for an undamped, open-loop structure, the predicted root shift $(d\lambda_n)_p$ is given by

$$(d\lambda_n)_p \approx \frac{1}{2} \sum_{a,r} C_{ar} \phi_{an} \phi_{rn} \tag{1}$$

where the coefficient matrix C_{ar} is a matrix of (damping) gains, and ϕ_{an}, ϕ_{rn} denote respectively the values of the nth mode shape at actuator station a and sensor station r.

Equation (1) may also be used to compute the unknown gains C_{ar} if the $d\lambda_n$ are considered to be desired root shifts or, equivalently, desired modal dampings. While an exact "inversion" of equation (1) does not generally exist, weighted least-squares type solutions can be devised to determine the actuator control gains C_{ar} necessary to produce the required modal damping ratios. This determination of the gains is the synthesis of LAC systems.

For structures which already have some damping or control systems in which sensor, actuator, or filter dynamics can either be ignored or are already embedded in the plant dynamics, the root perturbation techniques and cost function minimization methods above can similarly be used to synthesize low-authority controls.

2.13 Robustness of LAC Systems

When sensors and actuators are colocated (i.e. $a = r$), are complementary, and only rate feedback is used, formula (1) reduces to

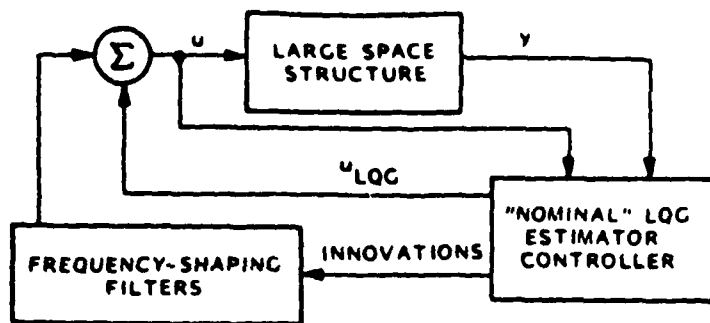$$d\lambda_n = -\xi_n\omega_n \approx \frac{1}{2} \sum_a C_a \phi_{an}^2$$

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Figure 10: LQG Control With Frequency-Shaping Filters.

which shows that the root shifts are always towards the left of the j-axis if all the gains are negative. This robustness result is obviously based on the assumption that both sensors and actuators have infinite bandwidth, and also that the structure was initially undamped. Several departures from this idealization occur in the actual practical implementation of the LAC systems. The most severe of these results from the finiteness of the actuators' bandwidths. More precisely, the second-order roll-off introduced by the actuator dynamics will always destabilize an undamped structure. However, when some natural damping is present in the structure, or when a passive damper is mounted in parallel with the actuator, additional active damping can be obtained without destabilizing the structure.

2.14 High-Authority Control Design

The HAC control design procedure can be based on any number of multivariable design methods, e.g. LQG, Q-parametrization, H_∞-optimization, etc. Increased penalties in the LQG cost functional are placed at those frequencies where less response is desired. The concept of frequency-shaped cost functionals was introduced prior to ACOSS [7].

The frequency shaping methods are useful in several areas of large space structures control. Three principal applications are important: (1) robustness (spillover avoidance), (2) disturbance rejection, and (3) state estimation.

2.15 Management of Spillover

Spillover in closed loop control of space structures is managed by injecting minimum control power at the natural frequencies of the unmodeled modes. Procedures for controlling spillover at high frequencies are usually discussed, although similar techniques are applicable for other regimes.

The high frequency spillover may be controlled by modifying the state or the control weighting. Conversion to the frequency domain gives the following performance index:

$$R(j\omega) = \left(\frac{(\omega^2 + \omega_0^2)}{\omega_0^2} \right) R$$

The problem of robustness (spillover management) is solved by making Q and R functions of frequency. Figure 10 depicts the modification to the nominal LQG controller. Observe that frequency shaping adds filters whose inputs are the innovation outputs of the state-estimator in the LQG controller.

2.16 Summary

The application of frequency-shaping methods to large space structures leads to a linear controller with memory. However, additional states are needed to represent frequency-dependent weights, hence, there is an increase in the controller order. The software needed for these controller designs is similar to that for standard LQG problems.
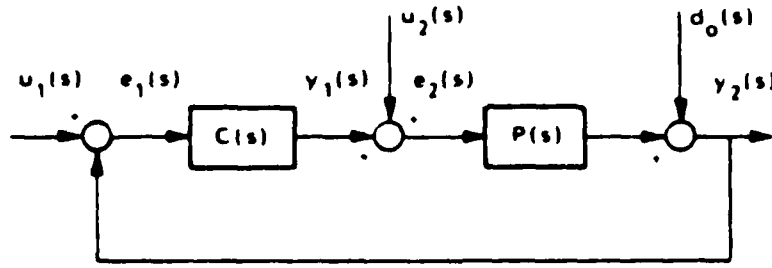
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Figure 11: The Unity-Feedback Structure.

3. Controller Design Using Q-Parametrization and H_∞ Optimization

During the last decade, mathematical theories of servo design have been based mainly on quadratic minimization of the Wiener-Hopf-Kalman type, usually applied to state-space models, e.g. LQG controls. However, despite the academic success of these methods, classical frequency response techniques relying on "lead-lag compensators" to reduce sensitivity have continued to dominate industrial servo design. One reason is that quadratic design tends to have poor sensitivity. On the other hand, the frequency domain description has proven to be more suitable to characterize uncertainties which arise in the plant approximation/identification, and frequency domain technique usually results in more robust design, e.g. frequency-shaped LQG can be viewed as an indirect frequency-domain design approach.

Two direct multivariable frequency domain design techniques have become popular in recent years: the Q-parametrization technique and the H_∞-optimal sensitivity.

3.1 Q-Parametrization Design

Consider the linear unity-feedback systems shown in Figure 11 where $P(s)$ is the given linear time-invariant plant. $C(s)$ is the linear compensator, u_1 is the reference input, u_2, and d_0 are respectively the plant-input disturbance and plant-output disturbance, and y_2 is the plant output.

The closed loop system input-output transfer function is given by

$$H_{yu} = \left[\begin{array}{ccc} C(I+PC)-1 & -C(I+PC)^{-1}P & -C(I+PC)^{-1} \\ PC(I+PC)-1 & P(I+CP)^{-1} & (I+PC)^{-1} \end{array} \right]$$

(For simplicity, we drop the argument s in $P(s), C(s)$ etc. in this section.)

By introducing the parameter (transfer function)

$$H_{yu} = \left[\begin{array}{c} u_1 \\ u_2 \\ d_o \end{array} \right] \rightarrow \left[\begin{array}{c} y_1 \\ y_2 \end{array} \right]$$

$$Q = C(I+PC)^{-1},$$

H_{yu} can be rewritten as

$$H_{yu} = \left[\begin{array}{ccc} Q & -QP & -Q \\ PQ & (I-PQ)P & I-PQ \end{array} \right]$$

Note that the closed loop input-output transfer function, for the given plant P, is completely specified by the parameter Q in a very simple manner: it involves only sums and products of P and Q.

In a typical control system design problem, the two most important closed loop transfer functions are $H_{y_2u_1}$ and $H_{y_2d_o}$: $H_{y_2u_1}$ is the transfer function from reference input u_1 to

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output y_2 and $H_{y_2 d_o}$ is the transfer function from plant-output disturbance d_o to output y_2. They specify respectively the servo-performance and regulator performance of the feedback system S. The two transfer functions are given by

$$H_{y_2 u_1} = PQ$$
$$H_{y_2 d_0} = I - PQ$$

and

Therefore the control design problem reduces to choosing the parameter Q so that the closed loop system S is stable and that $H_{y_2 u_1}$ and $H_{y_2 d_o}$ are "satisfactory". After the parameter Q is chosen, the corresponding compensator C can be obtained by the formula

$$C = Q(I - PQ)^{-1}$$

Hence, there is a one-to-one correspondence between C and Q. Consequently, for each parameter Q chosen, there is a unique compensator C which achieves the specified Q.

The selection of the parameter Q in the design process raises several questions: What are the conditions on Q so that the resulting compensator C is realizable (e.g. proper)? What is the class of all Q's which result in a stable feedback system? How is an "optimal" Q chosen?

Realizability: If the plant P is realizable, then the compensator C is realizable if and only if the parameter Q is realizable. Note that a physical plant is always realizable.

Global Parametrization: If the open loop plant P is stable, then the closed loop system S is stable if and only if Q is stable, since sums and products of stable transfer function matrices are stable. Consequently, the class of all stabilizing compensators is given by

$$\{Q(I - PQ)^{-1} | \ Q \text{ is stable}\}$$

and the class of all achievable stable input-output transfer matrix $H_{y_2 u_1}$ and the class of all achievable stable disturbance-to-output transfer matrix $H_{y_2 d_o}$ are given respectively by

$$\{PQ| \ Q \text{ is stable}\} \qquad \text{and}$$

$$\{I - PQ| \ Q \text{ is stable.}\}$$

These sets give global parametrization of all stabilizing compensators, and all achievable I/O characteristics in terms of a stable proper transfer matrix Q. In other words, the class of all "feasible" designs are parametrized by Q.

If the open loop plant P is not stable, additional constraints have to be added to the choice of Q, in addition to stability and realizability of Q. For example, Q must contain right half plane zeros to cancel the unstable poles of P. Currently, there are three approaches to obtain global parametrization of a given unstable plant: (i) Factorization representation theory [8]; (ii) Direct approach [9]; (iii) Two-step compensation [9].

Optimality: The Q-parametrization alone does not quantatively address the issue of optimal design. The designer selects Q from the class of "feasible" designs, on the basis of the desired input-output response, a priori knowledge of external disturbances, bandwidth, dynamic range and uncertainty of the plant, etc.

Optimal design based on the Q-parametrization and fractional representation framework has become very popular in the research community. The H_∞-optimal sensitivity design is among the results available.

3.2 H_∞-Optimal Sensitivity Design

The H_∞-optimal sensitivity design is an extension of the Q-parametrization technique to include a quantitative performance measure of the closed loop system and achievable optimality based on the performance measure. Roughly speaking, the H_∞ design problem is
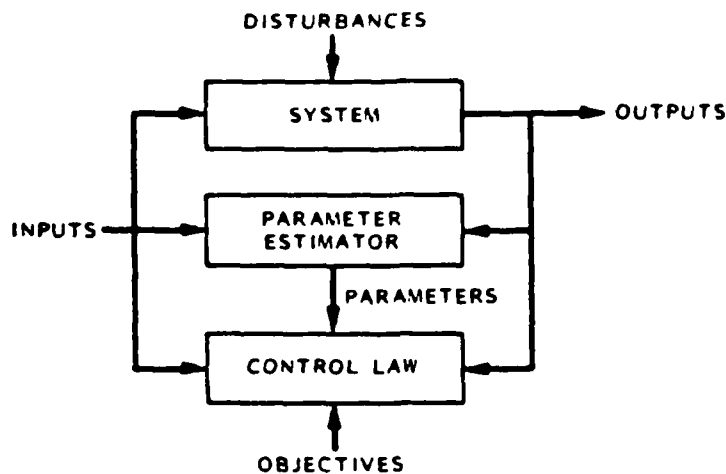
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Figure 12: Adaptive Control System.

the following: Given an open loop plant $P(s)$ and a low pass weighting function $W(s)$, find the compensator $C(s)$ so that the H_∞-norm of the weighted sensitivity $(I + PC)^{-1}W$ is minimized subject to the stability of the closed loop system.

Using the Q-parametrization formulation, the problem is equivalent to the following: Find a Q in H_∞ such that the closed loop system is stable and that $(I - PQ)W$ is minimized. Since the weighted sensitivity function is affine in Q, the equivalent problem is easier to solve than the original problem.

Solution to the H_∞-Optimal Sensitivity Problem: Based on the fractional representation (coprime factorization) formulation, several solutions have been proposed and algorithms given. However, all the proposed algorithms are conceptual in nature, suitable only for simple text book example. More effort is needed towards a numerically robust synthesis procedure.

4. Adaptive Control Techniques

Uncertainties in both disturbance spectra and system dynamical characteristics will limit the performance obtainable with fixed gain, fixed order controls. The use of adaptive type control, where disturbance and/or plant dynamics are identified prior to or during control, gives system designers more options for minimizing the risk in achieving performance benchmarks. For the case of LSS systems where performance levels are extremely high, it is absolutely necessary that disturbance and plant models be equally accurate. Since data from ground tests do not usually represent the flight condition accurately, it follows that an on-line procedure for identification and control is necessary.

The need to identify modal frequencies, for example, in high performance disturbance rejection systems has been shown in [1]. The deployment of high performance optical or RF systems may require on-line identification of critical modal parameters before full control authority can be exercised. Parameter sensitivity, manifested by performance degradation or loss of stability (poor robustness) may be effectively reduced by adaptive feedback mechanizations.

Most adaptive control algorithms can be described in the form shown in Figure 12. For example, one could select from the folowing catalogs of major areas:

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Model	Control Design	Adaptation
ARMAX	Model Reference	Gradient
State Space	Self-Tuning	Recursive Least Squares
	Pole-Placement	Recursive Max Likelihood
		Extended Kalman Filter

The schemes also differ in terms of update rates. Typically the outer control loop is at a fast rate, whereas the parameters from identification are updated more slowly. Adaptive schemes are referred to as *recursive* if the identification rate is a fixed multiple of the controller rate. If identification is used when necessary for calibration the scheme is referred to as *adaptive calibration*.

Although a great deal of research results are available about adaptive control and identification, unmodeled dynamics and broadband disturbances will significantly upset most algorithms.

4.1   Adaptive Calibration:

The use of a "slow" adaptive control, which is more practical than recursive adaptive control in most space applications is described in this section. It is referred to as a method of adaptive calibration. The term "slow" means that there is sufficient time to run batch identification before the control system is modified. The methodology provides a guaranteed level of performance given an "identified" model of the system together with the model error between the system and the identified model. In fact, the methodology generates performance versus model error tables (to be stored in the computer) from which the control design is immediately obtained. Moreover, the order of the control design is determined strictly on the basis of model error and performance demand.

4.2   Application of Adaptive Calibration:

The basic problem with control based on identified models is that without a measure of model error it is very easy to destabilize the system - particularly when the goal is high performance - as in LSS systems. Adaptive calibration is an approach which incorporates a measure of model error with robust control design in an iterative way so that identification is performed only where it is needed. A proposed adaptive calibration system is shown in Figure 13 with test results, using the CSDL #2 model, shown in Figure 14. The adaptive calibration procedure involves the following steps:

1. The model $M(s)$ is a 10-mode model which has been obtained from I/O data.

2. Estimate $\delta(\omega)$ = model error versus frequency using FFT. This is the dashed curve in Figure 15.

3. Using the identified model $M(s)$ and the model error $\delta(\omega)$, synthesize a robust control (section 2).

4. Calculate δ_{sm} - stability margin. This is the dashed curve in Figure 15. Compare to model error δ both plotted in Figure 16. If acceptable go to Step 6 and implement controller. Otherwise go to Step 5.

5. Modify filter windows, number of parameters (e.g. number of modes), or input spectrum and then repeat Step 1 to obtain new ID model. Figure 16 shows result of identification after one mode is added in the frequency domain region where the test fails.

6. Implement controller.
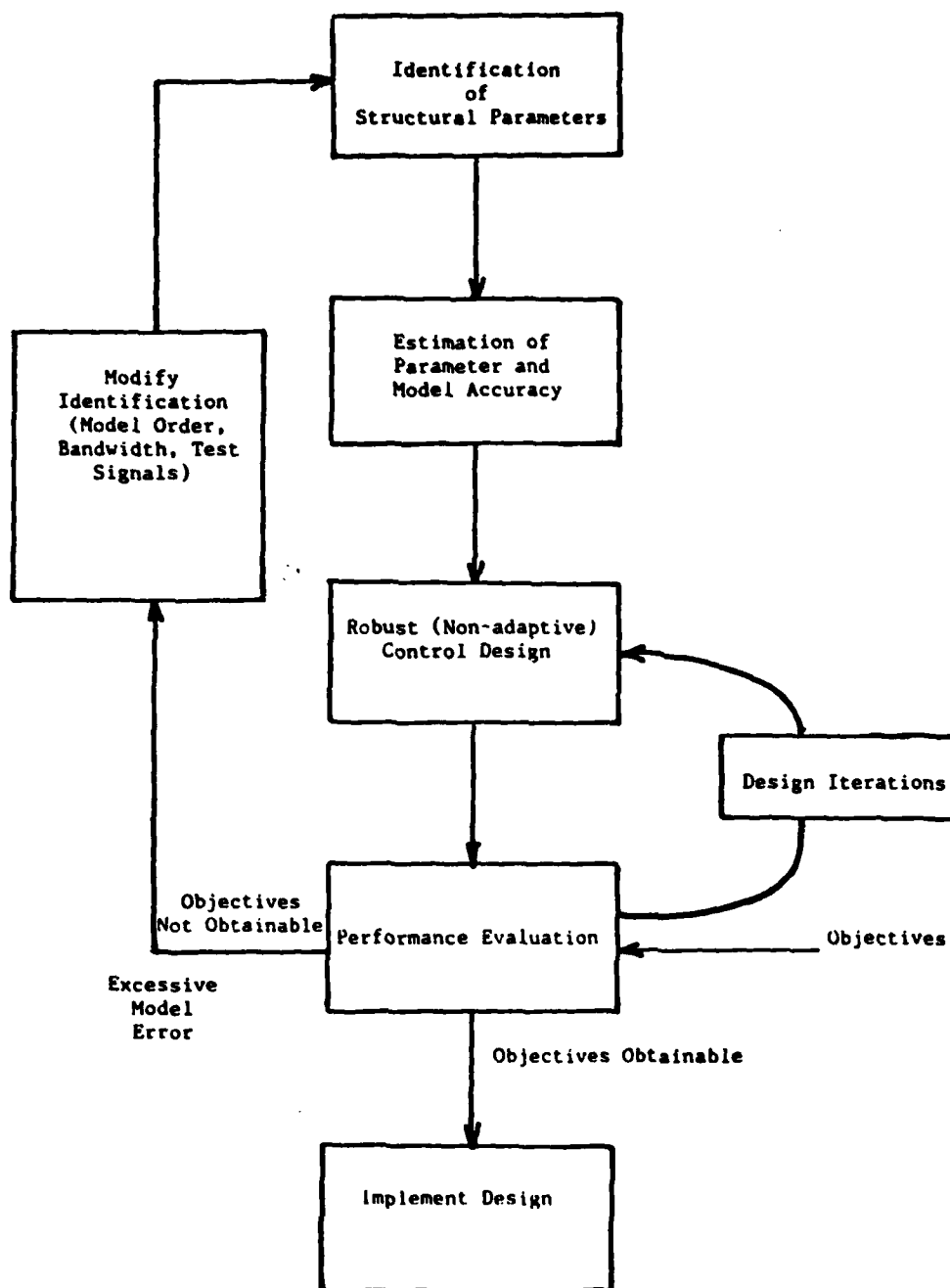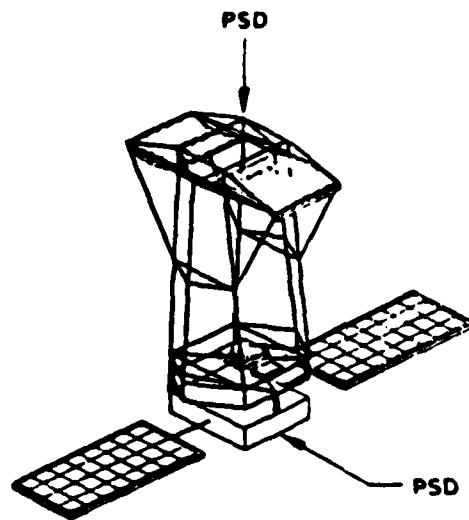
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Figure 13: An Adaptive Calibration System.

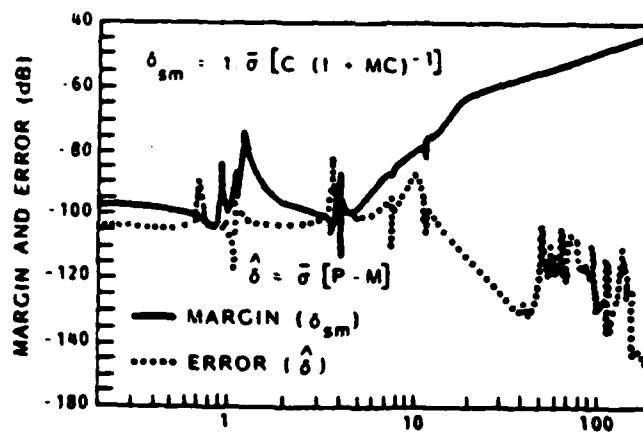CSDL MODEL NO. 2

Figure 14: Draper Simulation System.
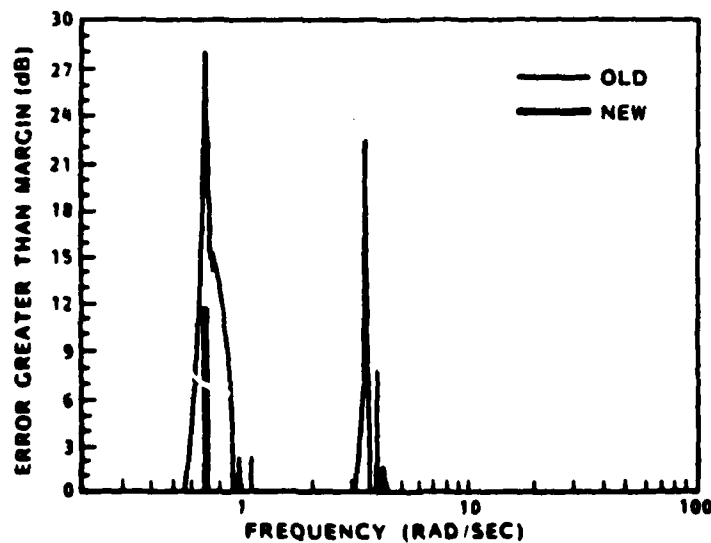

Figure 15: Comparison of Margin and Model Error.

Figure 16: Data Before and After ID Cycle Plots Show Where Error is Margin.

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Acknowledgments

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Methods of Averaging for Adaptive Systems

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Abstract

A summary of methods of averaging analysis is presented for continuous-time adaptive systems. The averaging results of Riedle and Kokotovic [1] and of Ljung [2] are examined and are shown to be closely related. Both approaches result in a sharp stability-instability boundary which can be tested in the frequency domain and interpreted as a signal dependent positivity condition.

1. Introduction

For a large class of adaptive systems, as well as for some output error identification schemes, a stability analysis in the neighborhood of the desired behavior leads to investigating the stability of the *linearized adaptive system* described by an equation of the form,

$$\dot{\theta} = -\epsilon z H(z'\theta) \tag{1}$$

where $\theta(t)\epsilon R^p$ is the adaptation parameter vector, $z(t)\epsilon R^p$ is the regressor, and $\epsilon > 0$ is the adaptation gain. The theory developed in [3,4] shows that the stability of adaptive systems in the neighborhood of the equilibrium trajectories is dependent on the stability of this system of linear time-varying equations. System (1) for example, can be obtained as a result of linearization of the adaptive system in the neighborhood of a "tuned" system, i.e., a system where the adaptive parameters are set to a constant value $\theta_*\epsilon R^p$ and whose behavior is deemed acceptable. Hence, in (1), $\theta(t)$ is the vector of parameter errors between the parameter estimate at time t and the tuned value θ_*, $z(t)$ is the regressor vector from the tuned system (e.g., filtered revisions of measured signals), and the scalar ϵ is the magnitude of the adaptation gain which essentially controls the rate of adaptation. The operator H *depends on the actual system being controlled or identified and also on the tuned parameter setting θ_*.*

It is shown in [3] that if the zero solution of (1) is uniformly asymptotically stable (u.a.s.), then the adaptive system is locally stable, i.e., the adaptive system behavior will remain in a neighborhood of the desired behavior provided the initial parameter error $\theta(0)$ and the effect of external disturbances are sufficiently small. Although these results were arrived at using input-output properties, local stability properties can also be obtained from the results on "total" stability [5].

1.1 Unmodeled Dynamics and Slow Adaptation

In the ideal case there are a sufficient number of adaptive parameters (the number p) such that the tuned parameter setting results in $H(s)$ being strictly positive real (SPR), i.e., $H(s)$ proper and stable, and $Re\ H(j\omega) > 0, \forall\omega\epsilon R_+$. Under these conditions, we have the following known results: (i) the zero solution of (1) is stable, i.e., $\theta(t)$ is bounded but not necessarily constant; (ii) if, in addition, $z(t)$ is persistently exciting, then the zero solution is u.a.s., thus, $\theta(t) \to 0$ exponentially fast as $t \to \infty$. The trouble starts when there are an

insufficient number of parameters to obtain $H(s)\epsilon SPR$, as is the case in adaptive control when the plant has unmodeled dynamics.

In this paper we will examine the stability of (1) when ϵ is small, $z(t)$ is persistently exciting, and $H(s)$ is not necessarily SPR but only stable. We will refer to this case as *slow adaptation*.

1.2 Approaches Based on Averaging

In a recent paper by Riedle and Kokotovic [1], a classical method of averaging as described by Hale [6] was applied to the linearized adaptive system. The result is a sharp stability-instability boundary determined by a signal dependent positivity condition which asserts that the zero solution of (1) is u.a.s. if

$$\lambda(\sum_{\omega \epsilon \Omega}[\alpha(\omega)\alpha(\omega)^*]Re\ H(j\omega)) > 0 \tag{2}$$

where Ω and $\{\alpha(\omega),\omega\epsilon\Omega\}$ are, respectively, the Fourier exponents and coefficients of $u(t)$. Condition (2) can be considered as a *signal dependent positivity condition*, but unlike the SPR condition $Re\ H(j\omega)$ is not required to be positive at *all* frequencies. Thus, this result is significantly weaker than the SPR condition required in the proof of stability of adaptive systems, e.g., [7,8]. In order to apply the averaging theory to obtain this result, the linearized system has first to be decoupled into slow (parameter) states and fast states. It is this transformation which is essential to the averaging approach and is a major contribution in the Riedle-Kokotovic method.

Averaging has also been applied to the counter-example of Rohrs et al. [9] by Åström [10,11]. In this analysis, by "freezing" the parameters, the parameter and state equations are decoupled thereby obtaining the asymptotic trajectories. Both of these averaging analyses assume that the system is periodic or almost periodic, an assumption that can be dispensed with by introducing the notion of a sample average [12].

In [13], the averaging approach is extended to nonlinear systems by introducing the *integral manifold* which completely separates the parameter and state equations. This latter approach is valid for the nonlinear adaptive system, and not just the linearized part. Related results can also be found in [14].

Averaging methods for adaptive systems have appeared in earlier work, the most notable of these being the averaging method developed by Ljung [2] for use in discrete-time recursive parameter estimation. The analysis shows that the convergence properties of the estimates can be determined from the stability properties of a related set of ordinary differential equations; the method usually referred to as the ODE analysis.

In this paper we summarize the results obtained by Riedle and Kokotovic [1,13] and show (heuristically) how they are related to the local stability analysis in [3,4] and the ODE averaging approach of Ljung in [2].

2. Adaptive Error System

Although it is unlikely that a truly generic adaptive error system can be formed to capture all the nuances of adaptive systems, the SISO adaptive system shown in Figure 1 is offered as a good representation for the purposes of analysis. The system equations are:

$$e = e_* - H_{ev}v \tag{3a}$$
$$z = z_* - H_{zv}v \tag{3b}$$
$$v = z'\theta \tag{3c}$$
$$\dot{\theta} = e\Gamma z \tag{3d}$$

The development of (3) can be found in [15,16] and in [17]. In (3), $e(t) \in \mathcal{R}$ is a measured error signal which drives the parameter update (3d), $z(t) \in \mathcal{R}^p$ is the regressor, and $\theta(t) \in \mathcal{R}^p$ is the parameter error between the current estimate at t and a *tuned* parameter setting $\theta_* \in \mathcal{R}^p$.

Figure 1: Adaptive error system

The selection of θ_* is based on complete knowledge of the actual plant and disturbances. The system corresponding to this setting is referred to as the *tuned system*. The signals $e_*(t) \in \mathcal{R}$ and $z_*(t) \in \mathcal{R}^p$ are outputs of the tuned system, and are referred to as the *tuned error* and *tuned regressor*, respectively. The signal $v(t) \in \mathcal{R}$ can be regarded as the adaptive control error.

The operators H_{ev} and H_{zv} are dependent on θ_* and describe how v effects the error and regressor signals. We assume here that H_{ev} and H_{zv} are linear-time-invariant (LTI) with stable proper transfer functions $H_{ev}(s)$ and $H_{zv}(s)$. This would arise, for example, when the plant to be controlled is LTI and the adaptive controller is linear in the adaptive parameters. The stability of H_{ev} and H_{zv} is a consequence of the definition of θ_* as the tuned parameter setting.

The operation Γ depends on the choice of parameter update algorithm. We will restrict attention here to the following representatives:

Gradient

$$(\Gamma z)(t) = \epsilon z(t)$$
$$\epsilon > 0 \tag{4}$$

Recursive Least Squares

$$(\Gamma z)(t) = P(t)z(t)$$
$$\frac{d}{dt}P^{-1}(t) = z(t)z(t)'$$
$$P(0) = P(0)' > 0 \tag{5}$$

3.   Global Stability and Passivity

It is of interest to determine under what conditions the adaptive error system (3,5) produces bounded outputs (θ, e, v, z) for *all* bounded initial parameter errors $\theta(o) \in \mathcal{R}^p$. This is what is meant here by "global" stability. As it turns out, it is possible to prove such a result provided that:

(i) $H_{ev}(s) \in SPR$ with gradient $\tag{6}$

(ii) $H_{ev}(s) - \frac{1}{2} \in SPR$ with least squares $\tag{7}$

(iii) $z_*, \dot{z}_* \in L_\infty^p$ and either $\tag{8}$

    a) $e_*, \dot{e}_* \in L_2 \cap L_\infty$ $\tag{9}$

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b) $e_*, \dot{e}_* \in L_\infty$ and
$$z \in PE \text{ (persistently exciting)} \tag{10}$$

Parameter convergence to a constant in R^p or to a well defined subset in R^p, requires that (9) be strengthened to:
$$e_*, \dot{e}_* \in L_2 \cap L_\infty, \quad z_* \in PE \tag{11}$$

The above results can be found in [7,15,16] and in [18]. Although of theoretical significance, they are not feasible to obtain in practice. In the first place, due to unmodeled dynamics [9], $H_{ev}(s) \in SPR$ is practically impossible to achieve in adaptive feedback and even in some output error identification. (This is not the case in equation error identification.) Secondly, when $e_*, \in L_\infty$ as in (10), it is required that $z \in PE$ which cannot be guaranteed in advance since z is *inside* the adaptive loop. Case (11) which requires $z_* \in PE$ - which is feasible to establish - conflicts with $e_*, \dot{e}_* \in L_2 \cap L_\infty$. The latter implies $e_*(t) \to 0$ which can only occur for $z_* \in PE$ - and where there are no unmodeled dynamics which we argue is not possible.

With these impossible to satisfy theoretical requirements, it is doubtful that a global stability theory can be attained which relies on passivity, i.e., condition (6,7). On the practical side, however, there is substantial evidence of well engineered algorithms that work without SPR [10]. These do not work for all $\theta(o)$ and for all e_*, z_* in L_∞, but rather, for restricted magnitudes and signal spectrums. For example, if $H_{ev}(s)$ is SPR for $\omega \leq \omega_{Bw}$ then it is expected that the adaptive system will be well behaved provided that there is insignificant excitation above ω_{Bw}. The following example illustrates some of this phenomena.

Example: Consider the model reference adaptive control (MRAC) system studied by Rohrs et al. [9] with plant
$$P(s) = \frac{2}{s+1} \frac{229}{(s+15)^2 + 4}$$

reference model
$$H_{ref}(s) = \frac{3}{s+3}$$

and adaptive control law
$$u = -\hat{\theta}_1 y + \hat{\theta}_2 r$$

The adaptive parameters are obtained from the gradient algorithm,
$$\dot{\hat{\theta}}_1 = ye$$
$$\dot{\hat{\theta}}_2 = -re$$
$$e = y - H_{ref}r$$

For this example we have the tuned error given by
$$e_* = H_{er}r$$

with
$$H_{er}(s) = \frac{458\theta_{*2}}{s^3 + 31s^2 + 259s + 229(1 + 2\theta_{*1})} - \frac{3}{s+3}$$

We also have
$$H_{ev}(s) = \frac{458}{s^3 + 31s^2 + 259s + 229(1 + 2\theta_{*1})}$$

Observe that $H_{er}(s)$ and $H_{ev}(s)$ are stable provided that
$$\theta_{*1} \in [0, 17.03)$$

Since $H_{ev}(s)$ has a relative degree of three, it follows that $H_{ev}(s)$ is not SPR, and so global stability is not guaranteed.

Figure 2 shows $\theta_1(t)$ vs. $\theta_2(t)$ for simulations corresponding to two selected inputs:
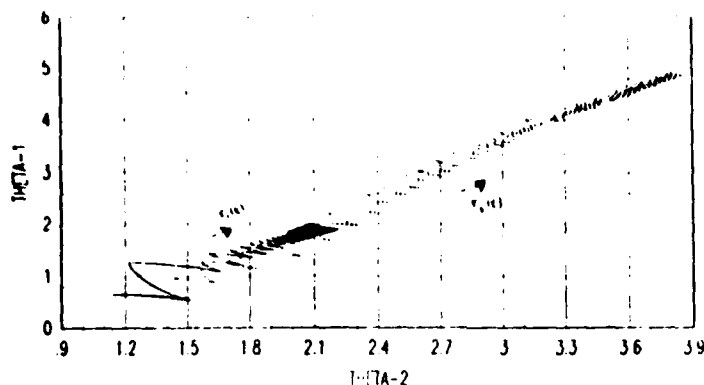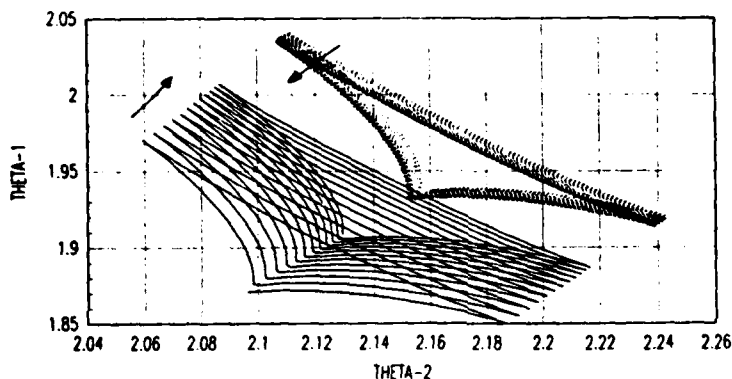
Figure 2: Parameter drift to inputs $R1$ and $R2$.

Figure 3: Blow-up of drift to input $R1$.

$$r_1(t) = 1 + sin\ 3t$$
$$r_2(t) = 1 + sin\ 5t$$

with initial conditions $\widehat{\theta}_1(0) = .65, \widehat{\theta}_2 = 1.15$ which satisfy the DC tracking requirement. The response to $r_1(t)$ undergoes a transient and then drifts down a line in $\mathcal{R}^2$ to an apparent stable orbit. Figure 2 shows a blow-up near the stable orbit as well as a trajectory which starts just below it and, drifts upward. The response to $r_2(t)$, however, is unstable in the sense that the parameters continue to drift and eventually $\widehat{\theta}_1(t)$ will exceed 17.03 and the system becomes unstable.

Most adaptive control systems show the characteristic behavior illustrated in our example. The parameter first exhibit a transient followed by a steady-state drifting. The papers by Åström [10,11] contain many examples. In the example here, the drifting appears to occur along a line in $\mathcal{R}^2$. In one case (input r_1) the drift stops and the parameters settle into a periodic orbit. With an apparently modest change in the input spectrum (r_2) the parameters now drift into the instability region. Therefore, either the orbital center has drastically changed and is now outside the constant parameter stability set, or else there is no stable orbit at all anywhere along the $\mathcal{R}^2$ line of drift.
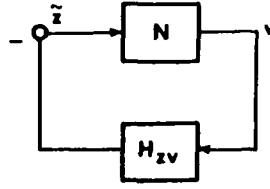
Figure 4: Equivalent feedback representation of adaptive error system.

In the forthcoming sections we will establish conditions under which the qualitative properties of the drifting phenomena can be predicted under slow adaptation. Our analysis is local and based on the classical methods of linearization and averaging for nonlinear systems.

4. Local Stability: Small Gain Theory and Averaging

Another way to view the system (3) under ideal conditions (6)-(11) is to arrange the system in the form shown in Fig. 4. Here, the forward path operator is defined by the map $N = \tilde{z} \rightarrow v$ such that

$$z = z_* + \tilde{z} \tag{12a}$$
$$\dot{\theta} = \epsilon[z e_* - z H_{ev}(z'\theta)] \tag{12b}$$
$$v = z'\theta \tag{12c}$$

with $\tilde{z}$ obtained from the feedback path,

$$\tilde{z} = -H_{zv}(\phi'\theta) \tag{13}$$

Observe that N is in effect the *linear* adaptive system that we mentioned earlier. Clearly $\tilde{z}$ is the amount by which the regressor $z(t)$ differs from the tuned regressor $z_*(t)$.

Now, let $T : \tilde{z} \rightarrow \tilde{\xi}$ denote the *loop-gain operator* defined by (12) and

$$\tilde{\xi} = -H_{zv}(\phi'\theta) \tag{14}$$

Small Gain Theory asserts [19] that if, for some $p \in [1, \infty]$, the L_p-gain of T is less than one, then the system is L_p-stable. Let $\gamma_p(T)$ denote the L_p-gain of T, i.e.,

$$\gamma_p(T) = inf\{k : \exists b \geq 0 \; s.t. \|T\tilde{z}\|_p \leq k\|\tilde{z}\|_p + b, \forall \tilde{z} \in L_p\} \tag{15}$$

when $p = 2$, it is possible to show that for all $\tilde{z} \in L_2$,

$$\|v\|_2 \leq c|\theta(0)| \tag{16}$$

where c is a constant *independent* of $\tilde{z}$ [16]. Hence,

$$\begin{aligned} \|T\tilde{z}\|_2 &= \|H_{zv}v\|_2 \\ &\leq \gamma_2(H_{zv})c|\theta(0)| \end{aligned} \tag{17}$$

Comparing this result with the definition of gain (15), we see that $\gamma_2(T) = 0$. Thus, under ideal conditions, *the loop-gain of the adaptive error system is zero!*

Now, suppose that $H_{ev}(s)$ is not SPR and $e_*(t) \neq 0$. One would expect that small deviations in the SPR'ness of $H_{ev}(s)$ and small non-zero magnitudes of $e_*(t)$ could be tolerated without trouble. Unfortunately, this is not quite the case. In the first place (16) holds without persistent excitation. This means that system N (12) is only uniformly stable (in the sense of Lyapunov). Recall that uniform stability is not robust to typical perturbations. Uniform asymptotic stability of N (equivalently exponential asymptotic stability, since N

is linear) is robust to a large class of perturbations. Thus, a basic idea behind the use of various forms of linearization theorems in the analysis of adaptive systems, is to insure that the system N is u.a.s. (uniformly asymptotically stable) which necessitates that $z_*(t)$ be persistently exciting. Since space limitations do not permit us to elaborate on linearization theory here, the interested reader is referred to [3,4]. We will, however, see that averaging imposes a natural linearization.

By restricting the magnitude of $\theta(o)$ and the magnitude and spectrum of $z_*(t)$ and $e_*(t)$, it is possible to obtain conditions to prove local stability [3,4]. The local stability property hinges on two premises: (i) the error system trajectories are in a (not necessarily small) neighborhood of the tuned solution, and (ii) the linear time varying system which maps $w \to \theta$ as given by

$$\dot{\theta} = -(\Gamma z_*)(t) H_{ev}(z_*'(\cdot)\theta(\cdot)) + (\Gamma w)(t) \tag{18}$$

is L_∞-stable, i.e., there exists constants k and b s.t. $\|\theta\|_\infty \le k\|w\|_\infty + b$. The choice of Γ comes from (4) or (5) and $z_*(t)$ is the tuned regressor. We can regard (18) as a linearization of the update algorithm. There are several ways to establish the L_∞ stability of (18).

4.1 Gradient Algorithm

We first consider the case when Γ represents the gradient algorithm, i.e., $(\Gamma z_*)(t) = \epsilon z_*(t)$ with $\epsilon > 0$.

In [20], it is shown that if $H_{ev}(s)\epsilon SPR$ and $z_*\epsilon PE$, then for all $\epsilon > 0, w \to \theta$ is exponentially stable, and hence, L_∞-stable. In [21], if $H_{ev}(s) = \overline{H}_{ev}(s) + s\Delta(s)$, $\overline{H}_{ev}(s)\epsilon SPR$, $\Delta(s)$ is stable, and $z_*\epsilon PE$ then for sufficiently small ϵ and $\|z_*\|_\infty, w \to \theta$ is still exponentially stable, and hence is L_∞-stable. This latter method relies on loop-transformations and application of small gain theory.

Another approach is to use averaging. In [1] it is shown that if $z_*\epsilon PE$ with the Fourier series representation

$$z_*(t) \sim \sum_k \alpha(\omega_k)e^{j\omega_k t} \tag{19}$$

and if the eigenvalues of the real matrix

$$B = \sum_k \alpha(\omega_k)\alpha(-\omega_k)' H_{ev}(-j\omega_k) \tag{20}$$

all have positive real parts, then for all sufficiently small $\epsilon > 0, w \to \theta$ is exponentially stable, and hence, L_∞-stable. Moreover, if any one eigenvalue of B has a negative real part, then $w \to \theta$ is exponentially unstable. Hence, there exists $w\epsilon L_\infty$ s.t. $|\theta(t)| \to \infty$ as $t \to \infty$ exponentially fast. It is obvious then when $H_{ev}(s)$ is *not SPR*, but only approximately so, then the Riedle-Kokotovic result provides a sharp stability-instability boundary. Note that when $H_{ev}(s)$ is SPR and $z_*\epsilon PE$ we have from [20] that $w \to \theta$ is exp. stable for *all* $\epsilon > 0$. At the present time, averaging theory as applied here, does not hold for all $\epsilon > 0$ even when $H_{ev}(s)$ is SPR. On the other hand, the result in [21] remains valid for $H_{ev}(s)\epsilon SPR(\Delta(s) = o)$ because then $\epsilon > 0$ is bounded above by infinity.

Example In this example we illustrate what happens when $Re\lambda(B) > 0$ but ϵ is too large. Consider the scalar system

$$\dot{\theta} = -\epsilon z_* H_{ev}(z_*\theta)$$

with $z_*(t) = sin(.35t)$ and $H_{ev}(s) = 1/(s^2 + 2s + 2)$. In this case B is a scalar and it is easily verified that $B > 0$. The simulations in Figure 5 with $\theta(0) = 1$ show that the zero solution is u.a.s. for $\epsilon = 4$ (and for all $\epsilon \le 4$), but is completely unstable for $\epsilon = 8$ (and for all $\epsilon > 8$).
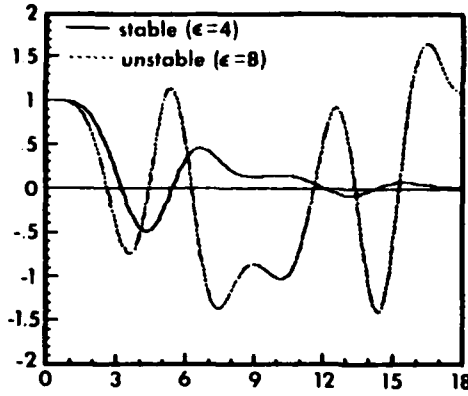
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Figure 5: Parameter trajectories with varying gain

4.2 Recursive Least Squares Algorithm

In this case we have from (5) that

$$(\Gamma z_*)(t) = P_*(t)z_*(t)$$
$$\frac{d}{dt}P_*(t)^{-1} = z_*(t)z_*(t)', P_*(0) > 0.$$

When $z_* \epsilon PE$ there exists $\alpha > 0$ such that

$$P_*(t)^{-1} = P_*(o)^{-1} + \int_o^t z_*(\tau)z_*(\tau)'d\tau \geq \alpha t \cdot I.$$

Thus, it is convenient to define $R(t) = \frac{1}{t}P_*(t)^{-1}$ for $t > 0$. Hence $R(t)^{-1} = tP_*(t) \leq \frac{1}{\alpha}I$ and we can write (18) and (5) as,

$$\dot{\theta} = \frac{1}{t}R^{-1}[w - z_* H_{ev}(z_*'\theta)]$$
$$\dot{R} = \frac{1}{t}(z_*z_*' - R) \tag{21}$$

When $H_{ev}(s) - \frac{1}{2}$ is not SPR we can now follow [2] and for $t \geq s$ and s sufficiently large, approximate the right hand side by its average. Letting "overbar" denote average (assuming it exists) we have:

$$\dot{\theta}(t) \approx \frac{1}{t}\overline{R}^{-1}(\overline{w} - (\overline{z_* H_{ev} z_*'})\overline{\theta})$$
$$\dot{\overline{R}}(t) \approx \frac{1}{t}(\overline{z_*z_*'} - \overline{R}) \tag{22}$$

Integrating from s to $s + T, T > 0$, gives

$$[\theta(s + T) - \theta(s)]/\int_s^{s+T} dt/t \approx \overline{R}^{-1}(w - (\overline{z_* H_{ev} z_*'})\overline{\theta}) \tag{23}$$

$$[R(s + T) - R(s)]/\int_s^{s+T} dt/t \approx \overline{z_* z_*'} - \overline{R} \tag{24}$$

Now change time scales $s + T \rightarrow \tau + \Delta\tau, \Delta\tau = \int_s^{s+T} dt/t$ and letting $s \rightarrow \infty$ gives the differential equations:

$$\dot{\theta}_A(\tau) = R_A(\tau)^{-1}[\overline{w} - B\theta_A(\tau)] \tag{25}$$

$$\dot{R}_A(\tau) = \overline{z_* z_*'} - R_A(\tau) \tag{26}$$

with $B = \overline{z_* H_{ev} z_*'}$ given by (20). These equations actually describe the asymptotic behavior of (18) in just the same way as they do for discrete-time [2]. In order to validate the approximations in each of the steps leading to (25) and (26), it is necessary to introduce

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various regularity conditions. A complete proof can be found in [2,22]. Here, as warned, we offer only heuristics.

Observe that in (26) as $\tau \to \infty$, $R_A(\tau) \to \overline{z_* z_*'}$. Thus, when $H_{ev}(s) - \frac{1}{2}$ is not SPR and $z_* \epsilon PE$ with Fourier representation (19) the asymptotes are stable if

$$Re\ \lambda(L) > 0 \tag{27}$$

where

$$
\begin{aligned}
L\cdot &= (\overline{z_* z_*'})^{-1} B \\
&= (\textstyle\sum_k \alpha(\omega_k)\alpha(-\omega_k)')^{-1} \sum_k \alpha(\omega_k)\alpha(\omega_k)' H_{ev}(-j\omega_k)
\end{aligned}
\tag{28}
$$

If $Re\ H(j\omega_k) \geq \rho > 0$ at low frequencies, and if $|\alpha(\omega_k)|$ is small at frequencies where $Re\ H(j\omega_k) \leq 0$, then $Re\ \lambda(L) = \rho$. Thus, all parameter asymptotes have a uniform rate of convergence which is not the case for the gradient algorithm with a time-invariant gain.

5. Averaging: A More General Approach

In this section we will establish a general form of the adaptive error system (3,5) which is useful for application of averaging methods. The first step is to transform (3,5) into a set of nonlinear time-varying differential equations. To do this observe that if $H_{ev}(s)$ and $H_{zv}(s)$ are strictly proper functions (a convenient illustrative, but not necessary, assumption) then we can write

$$
\begin{aligned}
H_{ev}(s) &= c'(sI - A)^{-1}b \\
H_{zv}(s) &= D(sI - A)^{-1}b
\end{aligned}
\tag{29}
$$

where $A \in R^{nxn}, b \in R^n, D \in R^{pxn}$, with $(A, b, [c\ D'])$ a minimal representation. Also, $Re\ \lambda(A) < 0$ reflecting the fact that H_{ev} and $H_{zv}(s)$ are stable. The error system (3) is then equivalently expressed as

$$
\begin{aligned}
e &= e_* - c'x \\
z &= z_* - D_z \\
\dot{x} &= Ax + bz'\theta \\
\dot{\theta} &= (\Gamma z)e
\end{aligned}
\tag{30}
$$

By eliminating the variables e and z we can reduce (30) to the coupled state-space description:

$$\dot{\phi} = \gamma(t)f(t, \phi, x) \tag{31}$$

$$\dot{x} = Ax + g(t, \phi, x) \tag{32}$$

With the gradient algorithm (4), let

$$
\begin{aligned}
\phi(t) &= \theta(t) \\
\gamma(t) &= \epsilon
\end{aligned}
\tag{33}
$$

and

$$
\begin{aligned}
f(t, \theta, x) &= z_*(t)e_*(t) - Q_*(t)x + c'xDx \\
Q_*(t) &= z_*(t)c' + e_*(t)D \\
g(t, \theta, x) &= b(z_*(t) - Dx)'\theta
\end{aligned}
\tag{34}
$$

With the recursive least squares algorithm (5), define:

$$R(t) = \frac{1}{t}P(t)^{-1} \tag{35}$$

and let

$$\phi(t) = \begin{pmatrix} \theta(t) \\ col\{R(t)\} \end{pmatrix}, \gamma(t) = \frac{1}{t} \tag{36}$$

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where the operator $col\{R\}$ stacks up the columns of the matrix R to form a vector. Thus,

$$f(t,\phi,x) = \begin{pmatrix} R^{-1}(z_*(t)e_*(t) & -Q_*(t)x + c'xDx) \\ col\{z_*(t)z_*(t)' & -z_*(t)(Dx)' - Dxz_*(t)' \\ & +Dx(Dx)' - R\} \end{pmatrix} \tag{37}$$

$$g(t,\phi,x) = b(z_*(t) - Dx)'\theta \tag{38}$$

The $col\{\cdot\}$ operator was used by Ljung in [2] to develop the discrete-time version of (31,32).

5.1 The Integral Manifold

The basic idea in the application of averaging methods to (31,32) is to see what happens when $\gamma(t)$ is small. Essentially, $\phi(t)$ slows down and we would expect to be able to approximate the right hand side of (31) with its average, i.e.,

$$\dot{\phi} \approx \gamma(t)\overline{f}(\phi) \tag{39}$$

where

$$\overline{f}(\phi) = \lim_{T\to\infty} \frac{1}{T} \int_o^T f(t,\phi,\overline{x}(t,\phi))dt \tag{40}$$

assuming the limit exists. (Such is the case, for example, when $f(t,\phi,x)$ and $g(t,\phi,x)$ are periodic in t for all bounded ϕ and x). The function $\overline{x}(t,\phi)$ is referred to as the state of the *frozen parameter* system, i.e., $\overline{x}(t,\phi)$ solves (32) whenever ϕ is a fixed vector. To emphasize this point we may express $\overline{x}(t,\phi)$ as the solution to the partial differential equation.

$$\frac{\partial}{\partial t}\overline{x} = A\overline{x} + g(t,\phi,\overline{x}), \overline{x}(0,\phi) = x(0)$$

The frozen parameter system was introduced in the averaging analysis proposed by Åström [11].

In order to remove the approximation in (39) we introduce the *integral manifold* as suggested by [13] [see [6] for discussion of the integral manifold]

The integral manifold M of (31,32) is the set,

$$M = \{t,\phi,x : x(t_o) = h(t_o,\phi(t_o)) \text{ implies } x(t) = h(t,\phi(t)), \forall t \geq t_o\} \tag{41}$$

By substituting $x = h(t,\phi)$ into (31,32), the *manifold function* $h(t,\phi)$ is seen to satisfy the partial differential equation

$$\frac{\partial h}{\partial t} + \gamma(t)\frac{\partial h}{\partial \phi}f(t,\phi,h) = Ah + g(t,\phi,h) \tag{42}$$

Whenever $\gamma(t)$ is sufficiently small, a reasonable approximation to $h(t,\phi)$ is given by $h_o(t,\phi)$ which is the solution to

$$\begin{aligned} \frac{\partial h_o}{\partial t} &= Ah_o + g(t,\phi,h_o) \\ &= F(\theta)h_o + G(\theta)z_*(t) \end{aligned} \tag{43}$$

where the last line follows from (34) with

$$F(\theta) = A - b\theta'D, \ G(\theta) = b\theta' \tag{44}$$

In (42), θ and t are regarded as independent variables and, hence, we can define the stabilizing parameter set

$$D_s = \{\theta \in \mathcal{R}^p : Re \ \lambda(F(\theta)) < 0\} \tag{45}$$

Thus, for $\gamma(t)$ sufficiently small, we can refer to $h(t,\phi)$ with $\theta\epsilon D_s$ as the *stable manifold*, which we will approximate by $h_o(t,\theta), \theta\epsilon D_s$.

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An important observation to make at this point is that the approximate manifold function $h_o(t, \phi)$ satisfies the same partial differential equation as the frozen parameter system state $\bar{x}(t, \phi)$. The only difference is in initial conditions. However, if $\theta \epsilon D_\bullet$ then as $t \to \infty$ we have $h_o(t, \phi) - \bar{x}(t, \phi) \to 0$ exponentially.

The final transformation on (31,32) is obtained by examining the behavior of (ϕ, x) in the neighborhood of the stable manifold. Introduce the error state,

$$\xi = x - h(t, \phi) \tag{46}$$

Using (46), and (31,32), we have

$$\dot{\phi} = \gamma(t) f(t, \phi, h(t, \phi) + \xi) \tag{47}$$

$$\dot{\xi} = F(\theta)\xi - \gamma(t)h_\phi(t, \phi)f(t, \phi, h(t, \phi) + \xi) \tag{48}$$

where we have used $h_\phi(t, \phi)$ to denote $\frac{\partial h}{\partial \phi}(t, \phi)$. If $\gamma(t)$ is sufficiently small it can be shown that under suitable regularity conditions we can approximate $h(t, \phi)$ with the frozen parameter state $\bar{x}(t, \phi)$ and obtain the approximate system,

$$\dot{\phi} = \gamma(t) f(t, \phi, \bar{x}(t, \phi) + \xi)$$

$$\dot{\xi} = F(\theta)\xi - \gamma(t)\bar{x}_\phi(t, \phi)f(t, \phi, \bar{x}(t, \phi) + \xi)$$

Moreover, if $\gamma(t)$ is sufficiently small and θ remains (moving slowly) in $D_\bullet$ then $\xi(t) \to 0$ exp. fast. As a result, by the same reasoning as in Section 4, the stability of the asymptotic system:

$$\dot{\phi}_A(\tau) = \overline{f}(\phi_A(\tau)) \tag{49}$$

where

$$\overline{f}(\phi) = \lim_{T \to \infty} \frac{1}{T} \int_o^T f(t, \phi, \bar{x}(t, \phi))dt \tag{50}$$

assuming the limit exists. The stability of (49) is given as follows. The proof is in [6]. ·
Theorem

Let ϕ^o denote a solution of

$$\overline{f}(\phi^o) = 0$$

and define the matrix,

$$G = \frac{\partial \overline{f}}{\partial \phi}(\phi^o)$$

Then, provided $Re\ \lambda(G) \neq 0$, the equilibrium solution $\phi_A(\tau) = \phi^o$ of the asymptotic system (49) is:

(i)   u.a.s. if $\max_i\ Re\ \lambda_i(G) < 0$.

(ii)   unstable if $\max_i\ Re\ \lambda_i(G) > 0$.

5.2   Application to Gradient Algorithm

Applying this result to (33,34) with the gradient algorithm and with $z_\bullet \epsilon PE$ and $\overline{z_\bullet e_\bullet} = 0$, gives $G = -B$ from (20). Since $\gamma(t) = \epsilon > 0$, we can only conclude that if $Re\ \lambda(B) > 0$ and ϵ is sufficiently small, then $\theta(t) \epsilon D_\bullet$ long enough for transients to die out, which is unprovable as yet in general.

Observe that $\theta_\bullet \in R^p$ such that $\overline{z_\bullet e_\bullet} = 0$ does not define an equilibrium of the actual system. All we can say is that with $\epsilon > 0$ small, there is a $\hat{\theta}(t)$ which orbits near (to order ϵ) the equilibrium of the asymptotic system. We can also choose to consider $\overline{z_\bullet e_\bullet} = 0$ as a defining equation in a *candidate* tuned setting. Other conditions would also have to hold (e.g. small $e_\bullet(t)$, etc.) which may be obtainable with proper input selection. In other words, the signals present *during adaptation* should be similar to those present during tracking or disturbance rejection. Otherwise, the algorithms choice of the tuned setting ($\overline{z_\bullet e_\bullet} = 0$) may be undesirable.

5.3 Application to Recursive Least Squares Algorithm

Under the same conditions and with the same provisions as above, $G = -L$ with L from (28). This time, since $\gamma(t) = 1/t \rightarrow 0$ as $t \rightarrow \infty$, we can conclude that if $Re\ \lambda(L) > 0$, then $\theta(t) \rightarrow 0$ as $t \rightarrow \infty$ at a rate $1/t$. In this case, due to the presence of $1/t$, the parameters $\hat{\theta}(t)$ asymptotically approach the solution of (49).

6. Concluding Remarks

The averaging theory described here, as well as averaging theory in general, has its uses and limitations for adaptive system. In the first place, the theory requires slow adaptation which can be counter-productive because performance can be below par for the long period of time it takes for the parameters to readjust. Secondly, averaging theory is a form of linearization so that the (nonlinear) adaptive system must be initialized in a (not necessarily small) neighborhood of the tuned system. On the positive side, however, we do obtain frequency domain conditions which explain the system behavior near the tuned solutions. In this sense, we can consider the results of averaging theory to be necessary conditions for good performance of adaptive systems.

To obtain the heralded goal of frequency-domain stability conditions, it may be inevitable to encounter linearization. Somewhat less intuitively appealing results can be obtained without resorting to direct linearization or averaging, e.g., in [4,17,21]. These results arise from a combination of small gain theory and perturbation theory.

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ADAPTIVE CALIBRATION: AN APPROACH TO UNCERTAINTY

MODELING AND ON-LINE ROBUST CONTROL DESIGN[*]

by

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Abstract

An approach is presented to the problem of on-line robust control design, referred to here as *adaptive calibration*. It is an iterative approach which modifies the filter and model structure characteristics used in methods of system identification involving the filtered prediction error. An estimate of model uncertainty is obtained which is then used to predict closed-loop system performance with the new control *if* it were implemented. If predicted performance does not meet the specified performance the filters and/or model structure are modified to enhance model accuracy in the frequency range required. An analysis is presented along with an illustrative example.

1. Introduction

Stringent closed-loop performance demands require very accurate models for controller design within the system bandwidth. Since the actual dynamics are not likely to be sufficiently like those obtained from testing methods, it is of practical importance to be able to identify the system dynamics on-line and then tune the controller to the updated model. The problem of on-line system identification and control tuning, referred to here as *adaptive calibration*, is a little like the story about Columbus: "He didn't know where he was going; when he arrived he thought he was someplace else; and when he returned he wasn't sure where he had been. And amazingly, he did it all with borrowed money!" The moral of the story is that in order to obtain a model from on-line identification which has the requisite accuracy, either unlike Columbus, we need to know where we are going, i.e., know the answer a priori, or else like Columbus, derive a means of calibration which will automatically adjust the identification accuracy, notwithstanding our ignorance of the true dynamics.

Although we use the term calibration to refer to control design based on an identified model, we are of course really discussing an adaptive controller. In this case, although we limit ourselves to infrequent controller adjustments, we ultimately face the same issues in the robustness of continuously changing adaptive controllers, e.g., Anderson et al. (1986).

In this paper we show how frequency domain bounds on the unmodeled dynamics of the identified model can be extracted from standard system identification procedures. Such bounds are required in order to evaluate the performance robustness of the control design, e.g., Doyle and Stein (1981), Safonov et al. (1981), and Vidyasagar (1985). The basis for the results here can be found in Ljung (1985), (1986) and Wahlberg and Ljung (1986), involving the use of parameter estimation methods of system identification as a means to estimate the system transfer function.

The paper is organized as follows: Section 2 discusses the calibration problem and the issues in robust control design. Section 3 addresses the problem of determining model uncertainty bounds from system identification methods. An example is presented in Section 4.

2. Calibration and Robust Control Design

In this section we discuss the general problem set-up for adaptive calibration in the context of disturbance attenuation. The system to be calibrated is shown in Fig. 1 and is described in discrete time by[*]

$$y(t) = G_0(q)u(t) + v(t) \qquad (2.1)$$

$$u(t) = -F_0(q)y(t) \qquad (2.2)$$

where $G_0(q)$ and $F_0(q)$ are the transfer functions[*] of system dynamics and feedback compensation, respectively. The function $v(t)$ represents the effective disturbances as seen at the output. We assume that $F_0(q)$ is a stabilizing controller for $G_0(q)$, but is to be replaced with another controller which is expected to improve performance. The controller $F_0(q)$ can be thought of as a controller from a previous calibration or as a "back-up" low-authority stabilizer. In a large space structure, for example, co-located rate dampers are used throughout the structure to provide robust stability, but these being of low-authority provide low performance.

In order to tune or replace the controller $F_0(q)$ in Fig. 1 we need a better model of (2.1) then what was used to design $F_0(q)$ in the first place. What we mean by the term *model* is:

(a) a *nominal* model of $G_0(q)$ and $v(t)$ in (2.1)

(b) a *set of uncertainty* associated with the nominal model. These together with

(c) a performance measure

constitute a *robust control design problem* for disturbance attenuation.

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[*] The variable q is used to denote the forward shift operator, i.e., $qx(t) = x(t+1)$.

[*] Strictly speaking, $G_0(q)$ is an operator whereas the complex function $G_0(z)$, $z \in C$, is a transfer function.

Performance Measure

Suppose that performance is specified in terms of average power, i.e., it is desired that

$$\text{avg}\{y^2(t)\} \triangleq \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} y^2(t) \leq y_{spec}^2 \qquad (2.3a)$$

and in addition that

$$\text{avg}\{u^2(t)\} \leq u_{spec}^2 \qquad (2.3b)$$

If $\Phi_y(\omega)$ and $\Phi_u(\omega)$ are the power spectral densities (PSD) of $y(t)$ and $u(t)$, then (2.3) is equivalent to*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_y(\omega) d\omega \leq y_{spec}^2 \qquad (2.4a)$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_u(\omega) d\omega \leq u_{spec}^2 \qquad (2.4b)$$

Note that ω is normalized frequency which is constrained to the interval $[-\pi, \pi]$. Let

$$u(t) = -F(q)y(t) \qquad (2.5)$$

be the feedback compensation which is to replace that in (2.2). If $\Phi_v(\omega)$ is the PSD of $v(t)$ then

$$\Phi_y(\omega) = |S(e^{j\omega})|^2 \Phi_v(\omega) \qquad (2.6a)$$

$$\Phi_u(\omega) = |Q(e^{j\omega})|^2 \Phi_v(\omega) \qquad (2.6b)$$

where

$$S(q) \triangleq [1 + G_0(q)F(q)]^{-1} \qquad (2.7a)$$

$$Q(q) \triangleq F(q)S(q) \qquad (2.7b)$$

For (2.6a) to be meaningful it is assumed that $v(t)$ has a PSD. *Unless otherwise stated, this is a standing assumption whenever we say that a signal has a PSD.*

Suppose that there exist functions $\Phi_y^{spec}(\omega)$ and $\Phi_u^{spec}(\omega)$ such that

$$\Phi_y(\omega) \leq \Phi_y^{spec}(\omega) \qquad (2.8)$$

$$\Phi_u(\omega) \leq \Phi_u^{spec}(\omega) \qquad (2.9)$$

are sufficient to satisfy (2.4)

Robust Control Design

If $G(q)$ and $\Phi_v(\omega)$ are perfectly known then the compensator $F(q)$ can be designed to satisfy (2.8) by any number of methods. Realistically, $G(q)$ and $\Phi_v(\omega)$ may vary or are not perfectly known. Suppose we have the set of uncertainty in $G_0(q)$ given by

$$G_0(q) = \hat{G}(q) + \Delta(q) \qquad (2.10)$$

where $\hat{G}(q)$ is a known nominal value whereas the only knowledge about $\Delta(q)$ is that it is stable and bounded by

$$|\Delta(e^{j\omega})| \leq \delta(\omega) \qquad (2.11)$$

In order to provide closed-loop stability for all uncertainty of the form (2.10), (2.11) it is necessary and sufficient that [see e.g., Callier and Desoer (1982)]

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* The PSD of $y(t)$ is defined as the two-sided Fourier transform of the autocorrelation function:

$$R_y(\tau) = \lim_{N \to \infty} \frac{1}{N} \sum_{t=1}^{N} y(t)y(t-\tau)$$

Note that the Fourier transform of $R_y(\tau)$ exists if $R_y(\tau)$ is in $l_1(\mathbb{R})$.

$$(i) \quad \hat{S}(q) \triangleq [1 + \hat{G}(q)F(q)]^{-1} \quad \text{is stable} \qquad (2.12a)$$

$$(ii) \quad \delta(\omega) < \delta_{sm}(\omega) \triangleq 1/|\hat{Q}(e^{j\omega})|, \; \forall \; \omega \in [-\pi,\pi] \qquad (2.12b)$$

$$(iii) \quad \hat{Q}(q) \triangleq F(q)\hat{S}(q) \quad \text{is stable} \qquad (2.12c)$$

The function $\delta_{sm}(\omega)$ in (2.12b) is referred to here as the *stability margin*. The sensitivity function (2.6) can be written as,

$$S(q) = \hat{S}(q)[1 + \Delta(q)\hat{Q}(q)]^{-1} \qquad (2.13)$$

This, together with (2.6), (2.11), gives the following frequency domain sufficient condition for (2.8) to hold:

$$\Phi_y(\omega) \leq \left[\frac{|\hat{S}(e^{j\omega})|}{1 - \delta(\omega)/\delta_{sm}(\omega)}\right]^2 \Phi_v(\omega) \leq \Phi_y^{spec}(\omega), \; \forall \; \omega \in [-\pi,\pi] \qquad (2.14a)$$

$$\Phi_u(\omega) \leq \left[\frac{|\hat{Q}(e^{j\omega})|}{1 - \delta(\omega)/\delta_{sm}(\omega)}\right]^2 \Phi_v(\omega) \leq \Phi_u^{spec}(\omega), \; \forall \; \omega \in [-\pi,\pi] \qquad (2.14b)$$

Observe that $\delta(\omega) < \delta_{sm}(\omega)$ is necessary for (2.14) to be satisfied, i.e., (2.14) implies stability robustness. Condition (2.14) can be thought of as a condition for *performance robustness*. That value of $\delta(\omega)$ for which equality holds in (2.14) is referred to as the *performance margin* and is given by

$$\delta_{pm}(\omega) = \delta_{sm}(\omega)(1 - \rho(\omega)) \qquad (2.15a)$$

where

$$\rho(\omega) \triangleq \max\left\{\frac{|\hat{S}(e^{j\omega})|[\Phi_v(\omega)]^{1/2}}{[\Phi_y^{spec}(\omega)]^{1/2}}, \frac{|\hat{Q}(e^{j\omega})|[\Phi_v(\omega)]^{1/2}}{[\Phi_u^{spec}(\omega)]^{1/2}}\right\} \qquad (2.15b)$$

Hence, (2.8) will hold if

$$\delta(\omega) \leq \delta_{pm}(\omega) \; , \quad \forall \; \omega \qquad (2.16)$$

Observe also that (2.16) makes sense only if the nominal closed-loop system strictly satisfies (2.8), i.e., if

$$\rho(\omega) < 1 \; , \quad \forall \; \omega \in [-\pi, \pi] \qquad (2.17)$$

Condition (2.16) can be used to evaluate candidate compensator designs $F(q)$. If the candidate satisfies (2.16) then the design can be implemented with confidence. If not, then (2.16) provides information as to the range of frequencies over which the design needs to be modified. Such information can be incorporated in the next design iteration, e.g., frequency-shaped LQG as described by Safonov (1981), Gupta (1980), and Stein and Athans (1985). In order to apply (2.16), it is necessary to have a nominal, dynamical model $\hat{G}(q)$, the model error bound $\delta(\omega)$, and the disturbance PSD $\Phi_v(\omega)$. It is clear that knowledge of $\Phi_v(\omega)$ can be relaxed to knowing an upper bound on $\Phi_v(\omega)$ in (2.16). The practical question to ask is how do we obtain this information? In terms of on-line tuning, can we do this automatically?

3. System Identification and Model Uncertainty

In this section we show how system identification methods -- specifically, parameter estimation -- can be used to provide a nominal model $\hat{G}(q)$, a model error bound $\delta(\omega)$, and an estimate of $\Phi_v(\omega)$.

Consider the least squares parameter estimator

$$\hat{\theta}_N = \underset{\theta \in D}{\arg} \; J_N(\theta) \qquad (3.1a)$$

$$J_N(\theta) = \frac{1}{N} \sum_{t=1}^{N} \epsilon_t^2 (t,\theta)dt \qquad (3.1b)$$

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where D is a subset of $\mathbb{R}^p$ and $\varepsilon_f(t,\theta)$ is the filtered error signal dependent on θ. Suppose $\varepsilon_f(t,\theta)$ has the PSD given by $\Phi_{\varepsilon_f}(\omega,\theta)$. Then,

$$\hat{\theta} \triangleq \lim_{N \to \infty} \hat{\theta}_N = \arg\min_{\theta \in D} J(\theta) \tag{3.2a}$$

$$J(\theta) = \text{avg}\{\varepsilon_f^2(t,\theta)\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_{\varepsilon_f}(\omega,\theta)d\omega \tag{3.2b}$$

The usefulness of (3.2) is that if N is large then $\hat{\theta}(N)$ is approximately $\hat{\theta}$. Moreover, recursive methods produce estimates which asymptotically approach $\hat{\theta}$, e.g., Ljung (1985), (1986).

Parametric Models

The results and notation of this section, which are needed in the sequel, are taken directly from Ljung (1985, 1986). Assume that $\varepsilon_f(t, \theta)$ is generated from a parametrized *observer* or *predictor* as follows:

$$\varepsilon_f(t, \theta) = L(q)\varepsilon(t, \theta) \tag{3.3}$$

where $L(q)$ is a stable filter operating on the *prediction error*

$$\varepsilon(t, \theta) = y(t) - \hat{y}(t, \theta) \tag{3.4}$$

Here $\hat{y}(t, \theta)$ is the one-step ahead prediction of $y(t)$ based on some parametric model for the data set $\{y(t), u(t)\}$ generated from (2.1), (2.2), i.e.,

$$y(t) = G_0(q)u(t) + v(t) \tag{3.5a}$$

$$u(t) = -F_0(q)y(t) \tag{3.5b}$$

where $v(t)$ is a zero-average sequence with PSD given by $\Phi_v(\omega)$. A typical parametric model is

$$y(t) = G(q, \theta)u(t) + H(q, \theta)e(t) \tag{3.6}$$

where G and H represent system dynamics and $e(t)$ is "unpredictable", e.g., a zero average sequence with constant PSD. Hence, we have

$$\hat{y}(t, \theta) = [1 - H^{-1}(q, \theta)]y(t) + H^{-1}(q, \theta)G(q, \theta)u(t) \tag{3.7a}$$

$$\varepsilon(t, \theta) = H^{-1}(q, \theta)[y(t) - G(q, \theta)u(t)] \tag{3.7b}$$

The parametrization of (3.6) can always be selected so that the map $(u, y) \to \hat{y}$ defined by (3.6) is stable. For example, consider the scalar ARMAX model,

$$A(q)y(t) = B(q)u(t) + C(q)e(t) \tag{3.8}$$

where A, B, C are polynomials in q^{-1} whose coefficients are the elements of θ. Hence,

$$G(q, \theta) = B(q)/A(q) \tag{3.9a}$$

$$H(q, \theta) = C(q)/A(q) \tag{3.9b}$$

then

$$\hat{y}(t, \theta) = \left[1 - \frac{A(q)}{C(q)}\right]y(t) + \frac{B(q)}{C(q)}u(t) \tag{3.10a}$$

$$\varepsilon(t, \theta) = \frac{A(q)}{C(q)}\left[y(t) - \frac{B(q)}{A(q)}u(t)\right] \tag{3.10b}$$

Observe that $(u, y) \to \hat{y}$ is stable if $1/C(q)$ is stable,[*] which is always possible to enforce. Hence, we take the set $D \subset \mathbb{R}^p$ in (3.1) as,

$$D = \{\theta \in \mathbb{R}^p : (u,y) \to \hat{y} \text{ in (3.6) is stable}\} \tag{3.11}$$

Note that when $C(q) = 1$ we have the *equation error* model, which together with (3.1) gives the least squares solution. In this case the set $D = \mathbb{R}^p$. Note also that filtering of the prediction error is equivalent to pre-filtering the data set, i.e., replace $\{y(t), u(t), t = 1, \cdots, N\}$ with $\{L(q)y(t), L(q)u(t), t = 1, \cdots, N\}$.

Frequency Domain Interpretation of Identification

The PSD of the filtered prediction error, denoted by $\Phi_{\varepsilon_f}(\omega, \theta)$ in (3.2) is given by:

$$\Phi_{\varepsilon_f}(\omega, \theta) = |L(e^{j\omega})|^2\Phi_\varepsilon(\omega, \theta) \tag{3.11a}$$

$$\Phi_\varepsilon(\omega, \theta) = |H^{-1}(e^{j\omega}, \theta)|^2[|\Delta(e^{j\omega}, \theta)|^2\Phi_u(\omega) + \Phi_v(\omega)] \tag{3.11b}$$

where $\Delta(q, \theta)$ is model error [see (2.10)], i.e.,

$$\Delta(q, \theta) = G_0(q) - G(q, \theta) \tag{3.12}$$

and $\Phi_v(\omega)$ is the PSD of $v(t)$. The above expressions are obtained by combining (3.6), (3.7), and (2.1), together with the following assumptions [see Ljung (1985, 1986)]:

$$\text{(A1) } G_0(q) \text{ stable} \tag{3.13}$$

$$\text{(A2) } F_0(q) = 0 \tag{3.14}$$

$$\text{(A3) } \Phi_{eu}(\omega) = 0 \tag{3.15}$$

Because the plant $G_0(q)$ is stable (A1), a stabilizing feedback is not required (A2). It is often the case that for practical reasons, a feedback is present regardless of the plant stability. The expression for $\Phi_\varepsilon(\omega, \theta)$ in (3.11b) is then,

$$\Phi_\varepsilon(\omega, \theta) = |H^{-1}(e^{j\omega}, \theta)|^2\{|\Delta(e^{j\omega}, \theta)|^2\Phi_u(\omega) + \Phi_v(\omega) + 2\text{Re}[\Delta(e^{j\omega}, \theta)\Phi_{vu}(\omega)]\}$$

In this paper we concentrate on the simple case when no stabilizing feedback is required, i.e., when assumptions (A1)-(A3) are satisfied.

Suppose we have determined $\hat{\theta} \in D$ from (3.2). We then have the transfer functions $G(q, \hat{\theta})$ and $H(q, \hat{\theta})$. Passing the data $\{y(t),u(t),t = 1,...\}$ through the predictor with $\theta = \hat{\theta}$ gives the prediction error

$$\varepsilon(t, \hat{\theta}) = H^{-1}(q, \hat{\theta})[y(t) - G(q, \hat{\theta})u(t)] \tag{3.16}$$

Hence, the PSD of $\varepsilon(t, \hat{\theta})$ is,

$$\Phi_\varepsilon(\omega, \hat{\theta}) = |H^{-1}(q, \hat{\theta})|^2[|\Delta(e^{j\omega}, \hat{\theta})|^2\Phi_u(\omega) + \Phi_v(\omega)] \tag{3.17}$$

In (2.14) we computed upper bounds on $\Phi_y(\omega)$ and $\Phi_u(\omega)$ which depends on knowledge of $|\Delta(e^{j\omega}, \hat{\theta})|$ and $\Phi_v(\omega)$. From (3.17) we see that it is precisely these functions which are the "unknowns," i.e., the function $H(q, \hat{\theta})$, $\Phi_\varepsilon(\omega, \hat{\theta})$, and $\Phi_u(\omega)$ can, in principal, he computed asymptotically from the data set $\{y(t), u(t), t = 1,...,N\}$ as $N \to \infty$. An interesting equivalent expression for (2.14) can be obtained by using (3.17) to eliminate $\Phi_v(\omega)$. By introducing the shorthand notation

$$\hat{\Phi}_\varepsilon(\omega) = \Phi_\varepsilon(\omega, \hat{\theta}) \ , \ \hat{G}(e^{j\omega}) = G(e^{j\omega}, \hat{\theta}) \ , \text{ etc.} \tag{3.18}$$

and by dropping the explicit ω and $e^{j\omega}$ dependence, we have from (3.17) that

$$\Phi_v = |\hat{H}|^2\hat{\Phi}_\varepsilon - |\hat{\Delta}|^2\Phi_u \tag{3.19}$$

Hence, the inequalities in (2.14) become,

$$\Phi_y \leq \Upsilon(|\hat{\Delta}|)|\hat{S}\hat{H}|^2\hat{\Phi}_\varepsilon \leq \Phi_y^{spec} \tag{3.20a}$$

$$\Phi_u \leq \Upsilon(|\hat{\Delta}|)|\hat{Q}\hat{H}|^2\hat{\Phi}_\varepsilon \leq \Phi_u^{spec} \tag{3.20b}$$

where

$$\Upsilon(\delta) \triangleq (1 - \delta/\delta_{sm})^{-2}[1 - (\delta/\delta_{bu})^2] \tag{3.21}$$

$$\delta_{bu} \triangleq |\hat{H}|(\hat{\Phi}_\varepsilon/\Phi_u)^{1/2} \tag{3.22}$$

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Recall from (2.12b) that

$$\delta_{sm} \triangleq 1/|\hat{Q}| \qquad (3.23)$$

Let Ω_0 and Ω_- denote the complementary frequency ranges defined by

$$\Omega_0 = \{\omega \in [-\pi, \pi] : \delta_{sm} > \delta_{hi}\} \qquad (3.24)$$

$$\Omega_- = \{\omega \in [-\pi, \pi] : \delta_{sm} \leq \delta_{hi}\} \qquad (3.25)$$

Thus,

$$\sup_{\delta \geq 0} Y(\delta) = \begin{cases} Y(\delta_{sm}) = +\infty & , \quad \forall \omega \in \Omega_- \\ Y(\delta_{pk}) < \infty & , \quad \forall \omega \in \Omega_0 \end{cases} \qquad (3.26)$$

where

$$\delta_{pk} = \delta_{hi}^2/\delta_{sm} \qquad (3.27a)$$

$$Y(\delta_{pk}) = \frac{1}{1 - (\delta_{hi}/\delta_{sm})^2} \qquad (3.27b)$$

Observe that $\forall \omega \in \Omega_-$, it is not possible to insure satisfaction of any *finite* requirement such as $\Phi_y(\omega) \leq \Phi_y^{spec}(\omega) < \infty$.

Suppose an upper bound $\Phi_v^{max}(\omega)$ on $\Phi_v(\omega)$ is available. Hence,

$$\Phi_v \leq \Phi_v^{max} \quad , \quad \forall \omega \qquad (3.28)$$

and it follows from (3.19) that

$$|\hat{\Delta}|^2 \in [\delta_{lo}, \delta_{hi}] \quad , \quad \forall \omega \in [-\pi, \pi] \qquad (3.29)$$

where

$$\delta_{lo} \triangleq [\delta_{hi}^2 - \Phi_v^{max}/\Phi_u]^{1/2} \quad , \quad \forall \omega \in [-\pi, \pi] \qquad (3.30)$$

Consequently, we have

$$\sup_{\delta} Y(\delta) = \begin{cases} Y(\delta_{sm}) = +\infty & , \quad \forall \omega \in \Omega_- \\ Y(\delta_{pk}) & , \quad \forall \omega \in \Omega_0^+ \\ Y(\delta_{lo}) & , \quad \forall \omega \in \Omega_0^- \end{cases} \qquad (3.31)$$

where

$$\Omega_0^+ = \{\omega \in \Omega_0 : \delta_{pk} > \delta_{lo}\} \qquad (3.32a)$$

$$\Omega_0^- = \{\omega \in \Omega_0 : \delta_{pk} \leq \delta_{lo}\} \qquad (3.32b)$$

A typical plot of $Y(\delta)$ vs δ for some $\omega \in \Omega_0^+$ is shown in Fig. 2.

The expression in (3.31) together with the performance requirement of (3.20) provide an indication as to the "goodness" of the identified model. Thus, at those frequencies where (3.20) fails, i.e., for $\omega \in \Omega_{bad}$ where

$$\Omega_{bad} = \{\omega \in [-\pi, \pi] : \sup_{\delta} Y(\delta) > \eta\} \qquad (3.33a)$$

$$\eta \triangleq \min\left\{\frac{\Phi_y^{spec}}{|\hat{Q}|\hat{h}^2\Phi_e}, \frac{\Phi_u^{spec}}{|\hat{S}|\hat{h}^2\Phi_e}\right\} \qquad (3.33b)$$

it is necessary to either abandon the specification or else perform the parameter estimation (3.2) under different conditions. For example, some choices are to modify the filter $L(q)$, change the parametric model order, or change the parametric model structure. Exactly what the rules are for such *re*-identification remains an open question, but certainly will rest on the experiment design criteria discussed in Ljung (1985, 1986), Wahlberg and Ljung (1986), and in Goodwin and Payne (1977).

For example, one possible way to alter the filter $L(q)$ is to enforce the condition:

$$|L_{new}| = \begin{cases} [\sup_{\delta} Y(\delta)/\eta]^{1/2}|L_{old}| & , \quad \omega \in \Omega_{bad} \\ |L_{old}| & , \quad \text{otherwise} \end{cases} \qquad (3.34)$$

Effect of Finite Data Record

Calculation of the bounds in (3.31) involve knowledge of $\Phi_e(\omega, \hat{\theta})$, $\Phi_u(\omega)$, $G(e^{j\omega}, \hat{\theta})$, etc. First of all, $\hat{\theta}$ from (3.2) is the asymptotic estimate. What is actually available is $\hat{\theta}_N$ from (3.1) for some finite N, e.g., typically on the order of $N = 1024$. Secondly, for any value of θ computation of $\Phi_e(\omega, \hat{\theta})$ involves the infinite data record $\{\varepsilon(t, \theta), t = 1,2,...\}$, whereas what is realistically available is $\{\varepsilon(t, \theta), t = 1,2,..., N\}$. The effect of finite data record on transfer function estimation accuracy has been examined in Ljung (1985, 1986), where the following approximation of $J_N(\theta)$ in (3.1b) is established for sufficiently large N:

$$J_N(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \|L(e^{j\omega})E_N(\omega, \theta)\|^2 \, d\omega \qquad (3.35a)$$

Also

$$E_N(\omega, \theta) \approx H^{-1}(e^{j\omega}, \theta)[Y_N(\omega) - G(e^{j\omega}, \theta)U_N(\omega)] \qquad (3.35b)$$

with E_N, Y_N and U_N the Discrete Fourier Transforms (DFT) of ε, y, and u, respectively.[*] We use the definition that

$$X_N(\omega) \triangleq DFT\{x(t)\} \triangleq \frac{1}{\sqrt{N}} \sum_{t=1}^{N} x(t)e^{-j\omega t} \qquad (3.36)$$

Observe that the DFT is actually computed at *discrete* frequencies, e.g., if N is even then

$$\omega = \frac{2\pi k}{N} \quad , \quad k = -\frac{N}{2} + 1, ..., \frac{N}{2} \qquad (3.37)$$

Based on the definition in (3.36), an estimate of the PSD of $x(t)$ given finite data $\{x(t), t = 1, ..., N\}$ is

$$\Phi_x^N(\omega) = \frac{1}{N} |X_N(\omega)|^2 \qquad (3.38)$$

This estimate of $\Phi_x(\omega)$ is not always smooth and often looks "noisey". Smoothing can be accomplished by introducing a *lag-window* which is effectively a frequency domain correlation of $|X_N(\omega)|^2$ with a weighting (or window) function $W(\omega)$ [see, e.g., Jenkins and Watts (1968) and Ljung and Glover (1981)].

A Calibration Procedure

The following steps, depicted in Fig. 3, are illustrative of a calibration procedure which follows more or less naturally from (3.31) and the parameter estimator of (3.2).

Step 1: Obtain $\hat{\theta}_N$ from (3.1) using the model structure (3.3)-(3.7).

Step 2: Using the identified system $G(q, \hat{\theta}_N)$, $H(q, \hat{\theta}_N)$ design a controller $u(t) = -F(q)y(t)$ satisfying (2.17).

Step 3: Using $\hat{\theta}_N$, filter the data set $\{y(t), u(t), t=1, ..., N\}$ to obtain the sequence $\varepsilon(t, \hat{\theta}_N)$ defined by (3.16).

---

[*] The approximations in (3.35b) follows from the fact that if $y(t) = T(q)u(t)$ then $\|Y_N(\omega) - T(e^{j\omega})U_N(\omega)\| \leq C/\sqrt{N}$ with $C = 2(\sup_{t \in [1,N]} \|u(t)\|)(\sum_{k=1}^{\infty} k\|T(k)\|)$, being the pulse response.

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Step 4: Using a PSD approximation estimate $\Phi_\varepsilon(\omega, \hat{\theta}_N)$ and $\Phi_u(\omega)$ from $\varepsilon(t, \hat{\theta}_N)$ and $u(t)$, respectively, and then calculate $\sup_\delta Y(\delta)$ vs. ω from (3.31).

Step 5: Check performance robustness using the test,

$$\sup_\delta Y(\delta) \le \eta \ , \quad \forall\ \omega \in |-\pi, \pi]$$

with η given by (3.33b). If the inequality holds for all ω, then implement the controller from Step 3. Otherwise, go to Step 6.

Step 6: At those frequencies where the test in Step 5 fails, either relax performance or reduce the model error by repeating Step 1. This involves changing the filter $L(q)$ or increasing the model order or both.

4. Application to a Laser Pointing Experiment

In this section we obtain the model error estimate $\delta_\Delta(\omega)$ of (3.24) using data from a laser pointing experiment. Analysis of this laser pointing experiment can be found in Walker, Shah, and Gupta (1984). As described there, the objective of the experiment (see Fig. 4) is to control the jitter of a laser beam. The single actuator consists of a proof-mass which exerts a reaction force on the flexible beam when the proof-mass is moved by an applied armature current input. A rate-sensor is provided on the actuator to measure actuator velocity (required when very high bandwidths are used). As the flexible beam vibrates, the laser beam changes its angular direction. A second sensor, a quad detector, mounted on the structural support picks up the position of the beam, as long as it is in its field-of-view. The laser beam strikes a mirror on the flexible member and then is reflected back by another mirror mounted on the proof-mass actuator. The resulting beam is split by a beam splitter into two rays, one going to the quad detector and the other going to a screen where the jitter is magnified. The proof-mass actuator controls both the flexible beam vibrations and the optical path tilt and hence can reduce the laser beam jitter. The mass of the actuator is greater than the flexible beam and therefore the interaction between the modes of the flexible beam and the actuator is significant.

Obtaining the Model for Control Design

Jitter control was desired in the region of 4 Hz to 20 Hz. Consequently, a sine sweep was applied to the actuator lasting about sixteen seconds, sweeping from 4 Hz to 20 Hz. The sampling rate was 51.2 Hz. Figure 5 shows that the magnitude of the input signal between 0 Hz and 4 Hz and between 20 Hz and 25.6 Hz is very low and therefore the model will not be accurate in those frequency ranges. Figure 6 shows a comparison with the "true" transfer function (the smooth dashed line) and a transfer function estimate obtained by simply taking ratios of the DFT's of the output and the input, i.e.,

$$G_N(\omega) = Y_N(\omega)/U_N(\omega) \qquad (4.1)$$

This latter estimate, referred to as the *empirical transfer function estimate* is very noisy as expected, since it is a modest reduction of the raw data. What we refer to here as the "true" system is an 8th order equation error model [(3.8) with $C(q) = 1$] obtained from the data by solving the least squares problem (3.1).

Using the raw data we now compute via (3.1) a 4th order equation error model. Its transfer function is plotted in Fig. 7

along with the empirical transfer function estimate. Figure 8 compares the 8th order "true" system (dark line) with the 4th order estimate (dashed line). The model error is obviously significant, particularly from about 3 to 9 Hz where control is critical.

We now compute $\delta_\Delta(\omega)$ from (3.22). In Fig. 9 we plot $\delta_\Delta(\omega)$ in comparison with the "true" model error between the 8th order model and the 4th order model, i.e., $|G(e^{j\omega}, \hat{\theta}_8) - G(e^{j\omega}, \hat{\theta}_4)|$. Except for some noise at the low and high frequencies, the estimate is very accurate where necessary. Therefore, it is possible to be confident about implementing a controller based on the model set $G(q, \hat{\theta}_4) + \Delta(q)$, $|\Delta(e^{j\omega})| \le \delta_\Delta(\omega)$. That is, if the test in Step 5 holds there is little chance of failure, whereas if the test fails one should proceed with caution before implementing a new controller. In the latter case it would be prudent to re-identify with filtered data, e.g., using the heuristic in (3.34). The results of such iterations are not explored in this paper.

5. Concluding Remarks

Preliminary results of calculating frequency-domain model error bounds from identified models has been presented. This calculation is critical if identified models are to be of use in control design. The results show that the proposed estimates are in good agreement with the true error in the frequency band that is critical for control design. Many issues remain, namely:

(1) What are the best ways to smooth the model error estimates, e.g., "windowed" PSD;

(2) What are the heuristics for iterating on the identification algorithm when the estimated model error is too large over some frequencies, e.g., increase model order and/or tune the data filters (these involve experimental design issues of input selection, etc.);

(3) What is the effect of structured uncertainty on the estimate of model error;

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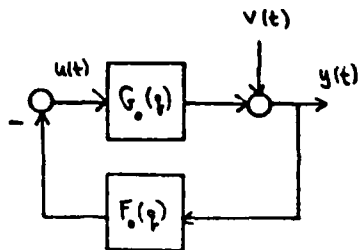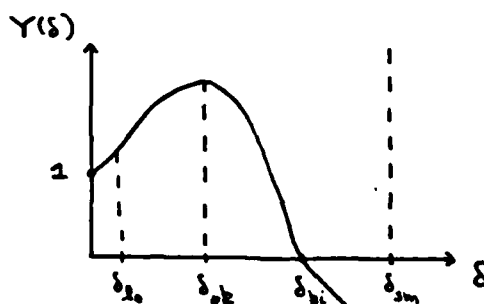
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Fig. 1 System to be calibrated

Fig. 2 Plot of $Y(\delta)$ vs. δ from (3.21) for some $\omega \in \Omega_o^T$ (3.31)
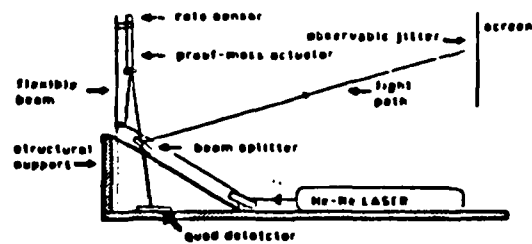

Fig 3 Flow chart of a calibration proceedure.

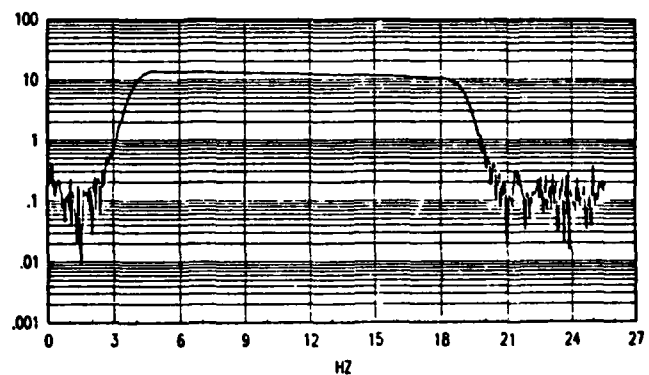Fig. 4 Laser pointing and jitter control experimental apparatus.

Fig.5a DFT of input u(t).

Fig.5b DFT of output y(t).

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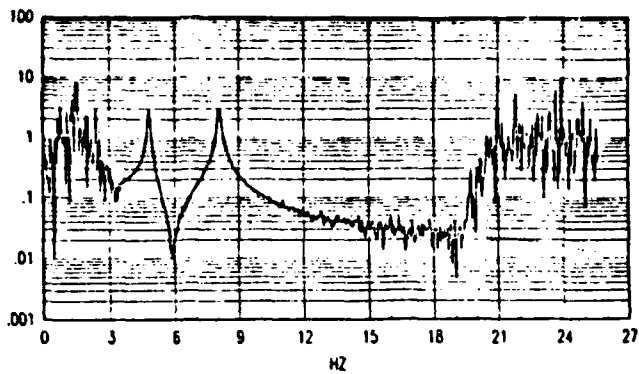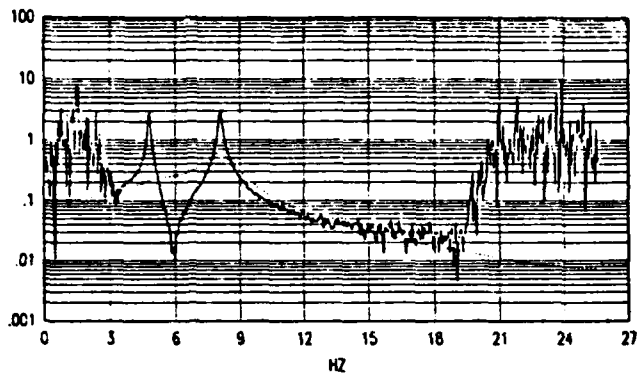
Fig. 6   Comparison of empirical transfer function with 8th order least squares model.

Fig. 8   Comparison of 8th order and 4th order least squares models.

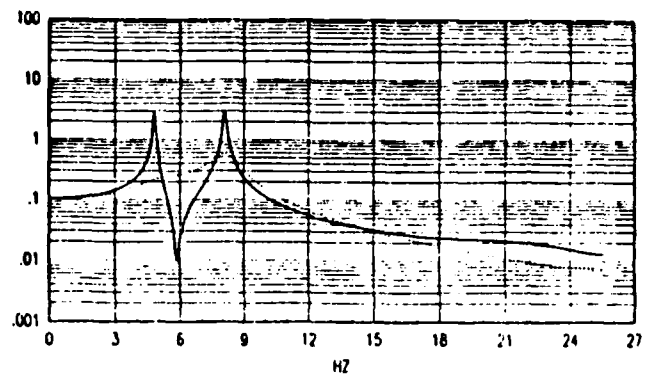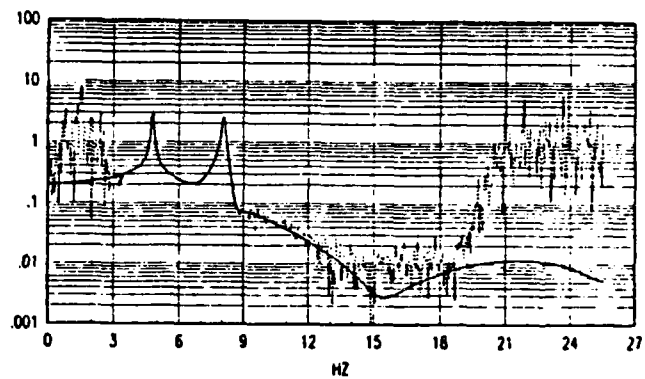Fig. 7   Comparison of empirical transfer function with 4th order least squares model.

Fig. 9.   Comparison of "true" model error and $\delta_{pi}(\omega)$ from (3.22) using PSD estimates with rectangular window (3.38).

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TRANSIENT ANALYSIS OF ADAPTIVE CONTROL

by

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ABSTRACT

Methods are developed to analyze the transient behavior of an adaptive system. It is shown that both small gain theory and the method of averaging can be used to predict some of the observed transient phenomena. An important tool in the analysis is fixed-point theory, illustrated by the Contraction Mapping Principal, which enables a sequential application of linear analysis to the separated state and parameter equations describing the adaptive system. It is also demonstrated that averaging theory applied on finite time intervals can predict transient phenomena without requiring slow adaptation.

1. INTRODUCTION

In this paper we consider an analysis of the transient response of adaptive control systems. An understanding of the transient is required in order to satisfy practical requirements such as those arising from constraints on tracking response and disturbance attenuation. For example, consider an adaptive system subject to abrupt set-point changes, e.g., step inputs. Typical system requirements are stated in terms of rise-time, overshoot, undershoot, and settling-time. Unlike a non-adaptive system, two sets of such requirements are needed; one set determined by the goal of the adaptive system, i.e., when the adaptive parameters are near convergences, and another set of requirements dealing with the transient, i.e., when the adaptive system is learning. The latter requirements include reasonable length of time for learning as well as bounds on responses imposed by hardware limitations.

Analysis of the adaptive systems transient will require sharper estimates of signal bounds and rate of convergence than currently exist. Consider the ideal case of perfect model matching, i.e., when there exists a *constant* unique setting of the adaptive parameters which produce zero error for all inputs. In this situation although it is possible to prove global stability and exponential parameter convergence, the system states can be arbitrarily large and the theory does not offer guidelines for adjustment, e.g., Goodwin and Sin (1984). Local stability analysis based on the method of averaging -- which is valid also in the non-ideal case -- provides some transient information but is restricted to parameter trajectories which vary slowly in a convex subset of the constant-parameter stability set, see e.g., Astrom (1983, 1984), Bodson et al. (1985), Riedle and Kokotovic (1986), and Anderson et al. (1986). Certainly one can argue that the latter is not restrictive in practical system tuning when the plant is slowly varying and initial parametrizations are close to a tuned setting. The drawback is that although the convergence rate is exponential, it is also very slow, whereas simulations show that onset of instability may produce very rapid learning, see e.g., Anderson (1983). Moreover, estimates of the convergence rate and the region of attraction obtained from the method of averaging can be quite conservative, and hence, do not provide a complete representation of the achievable transients leading to good performance, e.g., Mareels (1986).

In this paper we present an approach to the analysis of the transient based primarily on the use of fixed-point theorems, e.g., Kosut and Bitmead (1986). Here we discuss how the Contraction Mapping Principal can be used in conjunction with other methods of analysis including small gain, passivity, and averaging. Simulations will be provided in the final version of the paper which illustrate the use and limitations of the theory. The theory presented here is limited to a simple continuous-time gradient algorithm. Extensions as well as discrete time algorithms will be presented in the final version.

2. ADAPTIVE ERROR SYSTEM

A general structure for an adaptive system is shown in Fig. 1 and is described by the operator equations

$$\begin{bmatrix} e \\ \phi \end{bmatrix} = P \begin{bmatrix} w \\ u \end{bmatrix} = \begin{bmatrix} P_{ew} & P_{eu} \\ P_{\phi w} & P_{\phi u} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \tag{2.1}$$

$$u = -F(\theta)\phi \tag{2.2}$$

$$\theta = \Omega(\theta_0, e, \phi) \quad , \quad \theta(t_0) = \theta_0 \tag{2.3}$$

The adaptive system consists of three subsystems:

(1) the *plant subsystem P* which takes exogenous inputs w -- consisting of references and disturbances -- and the adaptive control inputs u into the *error e* and *regressor* ϕ;

(2) the *control subsystem* which transforms ϕ into u via the control matrix $F(\theta)$, which is parametrized by θ the adaptive parameter vector; and

(3) the *adaptation subsystem* Ω which uses the error signal e, the regressor signal ϕ, and the initial parameter value θ_0 to generate the parameter θ.
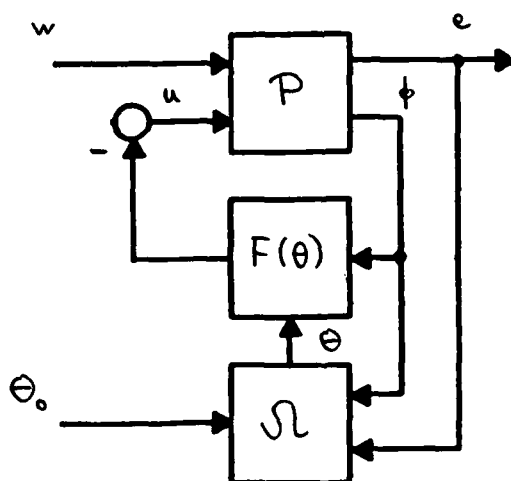

Fig. 1. Adaptive system

The structure of (2.1)-(2.3) can describe most of the standard forms of either continuous-time, discrete-time, or hybrid adaptive controller. Details on this structure can be found elsewhere, e.g., Kosut and Johnson (1983), Anderson et al. (1986). For example, a typical form for (2.2) is the bilinear structure

$$u = -\theta^T \phi \tag{2.4}$$

Typical forms for (2.3) include the simple continuous time gradient algorithm

$$\dot{\theta} = \varepsilon \phi e \tag{2.5}$$

or the discrete-time normalized gradient algorithm

$$\delta\theta = \varepsilon \frac{\phi e}{1 + \mu |\phi|^2} \tag{2.6}$$

where δ is the difference operator, i.e., $(\delta\theta)(t) = \theta(t) - \theta(t-1)$. In (2.5), (2.6), ε and μ are positive constants and $|\cdot|$ is the Euclidean norm, i.e., $|\phi| = (\phi^T\phi)^{1/2}$.

For illustrative purposes we will concentrate first on the continuous-time gradient algorithm (2.5). A convenient form for analysis is the adaptive *error system* which is formed by introducing the parameter error

$$\tilde{\theta}(t) = \theta(t) - \theta_* \tag{2.7}$$

We refer to θ_* as a *tuned* parameter, which is a constant vector of parameters producing desirable performance properties of (2.1), (2.4). When $\theta(t)$ is held fixed at θ_* the resulting system is referred to as the tuned system and is described by

$$\begin{bmatrix} e_* \\ \phi_* \end{bmatrix} = P \begin{bmatrix} w \\ u_* \end{bmatrix}$$

$$u_* = -F(\theta_*)\phi_* = -\theta_*^T\phi_* \tag{2.8}$$

The signals e_*, ϕ_*, and u_* are referred to as the tuned error, regressor, and control, respectively. If P is a linear-time-invariant (LTI) operator with transfer matrix $P(s)$, then it can be shown (Kosut and Friedlander, 1985), that the adaptive system (2.1), (2.4), and (2.5) can be described in *error form* by

$$\dot{\tilde{\theta}} = \varepsilon\phi e \quad , \quad \tilde{\theta}(t_0) = \tilde{\theta}_0$$

$$e = e_* - H_{ev}(\phi^T\tilde{\theta}) \tag{2.9}$$

$$\phi = \phi_* - H_{\phi v}(\phi^T \tilde{\theta})$$

where H_{ev}, $H_{\phi v}$ are stable LTI operators, dependent on θ_*, with transfer functions $H_{ev}(s)$, $H_{\phi v}(s)$, respectively. Stability of H_{ev}, $H_{\phi v}$ follows from the definition of the tuned parameter setting.

The system (2.9) can be shown to be globally stable, i.e., stable for all $\tilde{\theta}_0 \in \mathbb{R}^p$, provided that $H_{ev}(s)$ is SPR (strictly positive real), the tuned error $e_*(t)$ is zero, and $\phi_*(t)$ is bounded. Zero tuned error can be relaxed to $e_*(t)$ bounded and decaying exponentially fast to zero. Moreover, if $\phi_*(t)$ is persistently exciting then (2.9) is globally exponentially stable, i.e., stable for all $\tilde{\theta}_0 \in \mathbb{R}^p$. These results are typical, but not particularly useful for a transient analysis. In the first place the signal bounds are crude. Suppose for example that

$$H_{ev} = 1 \tag{2.10}$$

which is certainly SPR. Suppose also that $\exists$ constants $K > 0$ and $a > 0$ such that

$$| (H_{\phi v}v)(t) | \leq \int_0^t K e^{-a(t-\tau)} | v(\tau) | d\tau \tag{2.11}$$

We then have the following bounds:

(a) $\|\tilde{\theta}\|_\infty \leq | \tilde{\theta}_0 |$ (2.12a)

(b) $\|\phi^T \tilde{\theta}\|_2 \leq | \tilde{\theta}_0 |/(2\varepsilon)^{1/2}$ (2.12b)

(c) $\|\phi - \phi_*\|_2 \leq (K/a)| \tilde{\theta}_0 |/(2\varepsilon)^{1/2}$ (2.12c)

(d) $\|\phi - \phi_*\|_\infty \leq [K/(2a)^{1/2}]| \tilde{\theta}_0 |/(2\varepsilon)^{1/2}$ (2.12d)

If, in addition, $\phi_*(t)$ is *persistently exciting*, i.e., $\exists$ $T_0 > 0$, $\alpha_0 > 0$ such that

$$\lambda_{min}\left\{\frac{1}{T_0} \int_s^{s+T_0} \phi_*(t)\phi_*^T(t)dt\right\} \geq \alpha_0 , \quad \forall s \geq 0 \tag{2.13}$$

then $\tilde{\theta}(t) \to 0$ and $\phi(t) - \phi_*(t) \to 0$ exponentially with rate of convergence no slower than

$$\lambda = \frac{1}{2T_0} \ln\left[\frac{1}{1-\eta}\right] \tag{2.14}$$

where

$$\eta = \frac{2\tau T_0 \alpha_0}{(1+\varepsilon T_0 \beta_0)^2} \tag{2.15a}$$

$$\beta_0 = \|\phi_*\|_\infty^2 \tag{2.15b}$$

Hence, for small ε we have

$$\lambda \to \frac{1}{2} \varepsilon \alpha_0 \qquad , \quad \text{as } \varepsilon \to 0 \tag{2.16}$$

Observe also that for large ε the convergence rate decreases, i.e.,

$$\lambda \to \frac{\alpha_0}{2\varepsilon(\beta_0 T_0)^2} \qquad , \quad \text{as } \varepsilon \to \infty \tag{2.17}$$

Thus there is a limit to the convergence rate as seen by (2.16), (2.17).

The weakness of the bounds in (2.12) and the convergence rate estimate (2.14) is that they are conservative. It is often prudent, for example, to choose a small ε. By (2.12) we then have boundedness, but very large values can accrue since $\|\phi - \phi_*\|_\infty \sim (\varepsilon)^{-1/2}$, likewise the convergence rate estimate is very low. Simulations have verified this behavior in certain cases.

One may well ask the question: are these results intrinsic to the adaptive system or merely a result of the stability theory? The answer may ultimately turn out to be both, but for now we concentrate on the limitations of theory. It is obvious that when $H_{ev}(s)$ is not SPR the global theory breaks down completely. We turn then to local theory, e.g., the method of averaging. For example, from Anderson et al. (1986) it is shown that if

$$\text{Re } \lambda_{\min}\left\{ \lim_{T \to \infty} \frac{1}{T} \int_0^T \phi_*(t)(H_{ev}\phi_*)^T(t)dt \right\} > 0 \tag{2.18}$$

and $\bar\theta_0$ is in a convex subset of the constant parameter stability set, then for all small $\varepsilon > 0$,

$$\limsup_{t \to \infty} |\bar\theta(t)| = [1 + O(\varepsilon)]O(\|e_*\|_\infty) \tag{2.19}$$

Moreover, $\bar\theta(t)$ approaches the above limit set exponentially fast with a convergence rate of $O(\varepsilon)$. Thus, local stability is insured. We also have local *instability* insured

if any eigenvalue of the matrix in (2.18) has a negative real part.

The results in (2.18) and (2.19) are intuitively pleasing since they are robust, e.g., if $\mathrm{Re}H_{ev}(j\omega) > 0$ at the dominant frequencies of $\phi_*(t)$ then (2.15) will hold. Thus, H_{ev} can be "almost" SPR, at least where it counts. Also, the instability result keeps us from proposing kakimaymy excitation signals. The drawback, however, is that the bounds required on ε are conservative, and simulation results allow for much larger values, see e.g., Mareels et al. (1986). Secondly, if H_{ev} is SPR then we know from the global theory that ε need only be positive, yet the averaging result requires ε small. In the next few sections we hope to shed some light on these issues.

3. LINEAR ANALYSIS

We now consider only the prototypical ideal adaptive error system

$$\dot{\theta} = -\varepsilon\phi H(\phi^T\theta) \quad , \quad \theta(t_0) = \theta_0 \tag{3.1}$$

$$\phi = \phi_* - G(\phi^T\theta) \tag{3.2}$$

For convenience we have dropped all exogenous notation. We assume that the operators H and G are linear integral operators which are exponentially stable, i.e., as in (2.11) we have for some positive constants K and a that

$$| (Gu)(t) | \quad \text{or} \quad | (Hu)(t) | \leq \int_0^t Ke^{-a(t-\tau)} | u(\tau) | d\tau \tag{3.3}$$

Although system (3.1)-(3.3) does not at all represent all the myriad variants of adaptation mechanisms, we assert that any scheme for obtaining a transient analysis of this system will apply to the variants as well. Before describing some approaches, we begin with some examples which illustrate transient phenomena.

We start by examining (3.1) and (3.2) separately, as if they were decoupled *linear* systems. Consider first (3.1), and assume that $\phi(t)$ is a given function of time. If we take the scalar case with $H = 1$, (3.1) becomes,

$$\dot{\theta} = -\varepsilon\phi^2(t)\theta \quad , \quad \theta(0) = \theta_0 \tag{3.4}$$

Assuming $\phi(t)$ is our "input" the exact solution of (3.4) is

$$\theta(t) = \theta_0 \exp\left\{ -\varepsilon \int_0^t \phi^2(\tau)d\tau \right\} \tag{3.5}$$

It is clear that $|\theta(t)| \leq |\theta_0|$ as stated in (2.12a). Suppose $\phi(t)$ is given by

$$\phi(t) = ae^{-bt} + \sin ct \tag{3.6}$$

where b and c are positive constants. Suppose that for $t \in [0, t_1]$, $ae^{-bt} \approx a \gg 1$. Then

$$\phi(t) \approx a \quad , \quad t \in [0, t_1] \tag{3.7}$$

We then have

$$\theta(t) \approx \theta_0 \exp\{ -\varepsilon a^2 t \} \quad , \quad t \in [0, t_1] \tag{3.8}$$

As time goes on $\phi(t) \to \sin ct$ and hence

$$\theta(t) \to \theta_0 \exp\left\{ -\frac{\varepsilon}{2} t - \frac{\varepsilon}{4c} \sin 2ct \right\} \tag{3.9}$$

Thus, $\theta(t) \to 0$ exponentially as expected, since (3.6) is (asymptotically) persistently exciting for (3.4). Observe, however, that the early convergence rate from (3.8) is εa^2 which may be considerably larger than the final convergence rate of $\varepsilon/2$ from (3.9). It is precisely this kind of transient phenomena which needs to be addressed.

Now consider (3.2) separately and suppose that

$$G(s) = \frac{1}{s+a} \quad , \quad a > 0 \tag{3.10}$$

and that

$$\theta(t) = -re^{-\lambda t} \quad , \quad a > \lambda > 0 \tag{3.11}$$

System (3.2) becomes

$$\phi - \phi_* = -x$$

$$\dot{x} = -(a + \theta(t))x + \theta(t)\phi_*(t) \quad , \quad x(0) = 0 \tag{3.12}$$

The solution $x(t)$ is then explicitly given by:

$$x(t) = \int_0^t F(t, \tau)\theta(\tau)\phi_*(\tau)d\tau$$

$$F(t, \tau) = \exp\left\{ -\int_\tau^t (a + \theta(s))ds \right\} \tag{3.13}$$

It is clear that $x(t) \to 0$ exponentially. However, $x(t)$ may obtain quite a large peak value, particularly if $\lambda \ll a$. Recall that $\lambda = O(\varepsilon)$ as seen from (2.15)-(2.17). In fact when $\lambda \ll a$ the initial behavior of (3.12) is like

$$\dot{x} = -(a - r)x - r\phi_*(t) \tag{3.14}$$

If $a - r < 0$, or even near zero, then $x(t)$ will initially grow until $a - re^{-\lambda t}$ becomes negative. In fact the time to reach the peak in $x(t)$ is approximately

$$t_{pk} \approx \frac{1}{\lambda} \ln(r/a) \tag{3.15}$$

Hence, the "more" initial instability $(r > a)$ the longer it takes to turn the system around and the bigger will be the value of $|x(t_{pk})|$. [The full paper will contain simulations of those phenomena.]

We now discuss analysis techniques which can predict the demonstrated transient behavior for (3.1) and (3.2) separately. In the next section we will see that these separate linear analysis can be joined by applying fixed point theory.

For systems (3.1) consider the interval $[t_0, t_0 + T]$. Define the *sample average* matrix

$$R = \frac{1}{T} \int_{t_0}^{t_0+T} \phi(t)\phi^T(t)dt \tag{3.16}$$

We then have that

$$\theta^T(t_0 + T)\theta(t_0+T) \leq \theta^T(t_0)[I - \eta R]\theta(t_0) \tag{3.17}$$

where

$$\eta = \frac{2\varepsilon T}{(1+\varepsilon\beta T)^2} \tag{3.18}$$

with

$$\beta = \sup_{t \in [t_0, T]} |\phi(t)|^2 \tag{3.19}$$

The proof of (3.17)-(3.19) is omitted here but follows directly by using the Lyapunov function $V(t) = |\theta(t)|^2$ and then differentiating along (3.1). Observe that for all $\varepsilon > 0$, we have $\eta \in (0, 1)$ and also that

$$|I - \eta R| \leq 1 \tag{3.20}$$

which verifies (2.12a). It is usually the case that R is at worst rank deficient, i.e., $1 \leq \text{rank}(R) \leq \dim(\theta)$, in which case it can be shown a submatrix of $I - \eta R$ of order $\text{rank}(R)$ is contractive, i.e., a linear combination of θ parameters is contracting over $[t_0, t_0 + T]$. A similar result can be obtained for the system

$$\dot{\theta} = - \varepsilon \phi H(\phi^T \theta) \tag{3.21}$$

where H is not necessarily SPR but can be expressed as $H = \overline{H} + \tilde{H}$ where $\overline{H}$ is SPR and $\tilde{H}$ is "small". The result is analogous to (3.17)-(3.20) but with the exception that $\varepsilon \|\tilde{H}\|$ is required to be sufficiently small and R is given by

$$R = \frac{1}{2T} \int_{t_0}^{t_0+T} (H\,\phi)(H\,\phi)^T dt \tag{3.22}$$

An earlier version of this result is in Anderson et al. (1984).

Now consider system (3.2) as if $\theta(t)$ had given properties. It is convenient to define the regressor error

$$\tilde{\phi}(t) = \phi(t) - \phi_*(t) \tag{3.23}$$

We then have (3.2) written as

$$\dot{\tilde{\phi}} = - G(\theta^T(\phi_* + \tilde{\phi})) \tag{3.24}$$

Hence,

$$\tilde{\phi} = - Q(\theta^T \phi_*) \tag{3.25}$$

where Q is the linear integral operator

$$Q = (I + G\theta^T)^{-1}G \tag{3.26}$$

whose kernal satisfies

$$Q(t, \tau) = G(t, \tau) - \int_{\tau}^{t} G(t, s)\theta^T(s)Q(s, \tau)ds \tag{3.27}$$

It follows from (3.3) that if

$$|\theta(t)| \leq re^{-\lambda t} \quad , \quad \lambda > 0 \tag{3.28}$$

then

$$| \tilde{\phi}(t) | \leq K \| \phi_* \|_\infty \ \frac{re^{Kr/\lambda}}{a - \lambda} \ (e^{-\lambda t} - e^{-at}) \tag{3.29}$$

which may be obtained from (3.3) and (3.28) by applying the Bellman-Gronwall Lemma. If $a > > \lambda$ then the peak value of $| \tilde{\phi}(t) |$ could be as large as, but no larger than,

$$\| \tilde{\phi} \|_\infty \leq K \| \phi_* \|_\infty \ \frac{r}{a} \ a^{Kr/\lambda} \tag{3.30}$$

For small λ the term $e^{Kr/\lambda}$ can be quite big unless of course r is small, i.e., $\theta(t)$ is small to begin with. From the previous example we saw from (3.8), (3.9) that even though λ may be of order ε ultimately (3.9), it is possible during certain transients for λ to be large, e.g. (3.8) Further examples will be presented in the final paper.

4. NONLINEAR ANALYSIS

4.1 Fixed Point Theory

In this section we show how the linear analyses of Section 3, separately applied to (3.1) and (3.2), can be brought together. This is accomplished by application of the Banach Fixed Point Theorem (FPT), i.e., the Contraction Mapping Principal (CMP). We need the following definitions.

If M is a subset of a Banach space B with norm $\| \cdot \|$, and Γ is an operator mapping $M \rightarrow B$, then Γ is a *contraction on* M if $\exists$ constant $\sigma \in [0, 1)$ such that

$$\| \Gamma x - \Gamma y \| \leq \sigma \| x - y \| \quad , \quad \forall \ x, y \in M \tag{4.1}$$

The constant σ is the *contraction constant for* Γ on M. A *fixed point* of $\Gamma : M \rightarrow M$ is a point (function) $x \in M$ such that $x = \Gamma x$. We now have the following theorem as stated in Hale (1969).

> **Contraction Mapping Principal (CMP):** If M is a closed
> subset of a Banach space B and $\Gamma : M \rightarrow M$ is a contraction
> on M, then Γ has a unique fixed point in M.

In order to apply the CMP to the adaptive system (3.1), (3.2) we need to identify the operator Γ and the space M. For example, consider the operator Γ defined by

$$\Gamma: \overline{\theta} \to \theta : \begin{cases} \phi = \phi_* - G(\overline{\theta}^T \phi) & \text{(4.2a)} \\[2ex] \dot{\theta} = -\varepsilon\phi H(\phi^T\theta) \quad , \quad \theta(0) = \theta_0 & \text{(4.2b)} \end{cases}$$

It is clear that fixed points of Γ are solutions to the adaptive system, i.e., $\theta = \Gamma\theta$ is equivalent to (3.1)-(3.3). In fact, the solution to (3.1)-(3.3) is unique and is also a fixed point of Γ. A convenient choice for the space M is

$$M = \left\{ \theta \in C[0, T] : \|\theta\|_M \le r \right\} \tag{4.3}$$

where we choose the norm on M as

$$\|\theta\|_M = \sup_{t \in [0, T]} | e^{\lambda t}\theta(t) | \tag{4.4}$$

with λ and r chosen positive constants. We proceed to apply the CMP as follows. First, we establish that Γ maps $M \to M$. Hence, pick any $\overline{\theta} \in M$, i.e., $| \overline{\theta}(t) | \le re^{-\lambda t}$, and solve (4.2a). With $\overline{\theta}(t)$ so chosen (4.2a) is a linear system and the analysis in Section 3 applies. In particular, (3.29) holds. We proceed to (4.2b) which is again linear since $\phi(t)$ is obtained from (4.2a) and is not dependent on $\theta(t)$. Thus, the results of (3.16)-(3.20) apply. We then enforce $\|\theta\|_M \le r$, which restricts the relation among θ_0, K, a, λ, and r so that $\Gamma = M \to M$. In this case we also obtain that Γ is contractive on M under the same conditions causing $\Gamma: M \to M$. These conditions will be stated in the full version of the paper.

The choice of M in (4.3) leaves too much imprecision in $\theta(t)$. A "finer" choice is to select the norm on M to be,

$$\|\theta\|_M = \sup_{t \in [0, T]} | e^{tB}\theta(t) | \tag{4.5}$$

where $B \in \mathbb{R}^{p \times p}$ with $\text{Re } \lambda(B) \ge 0$. This allows for greater possibilities in the transient of $\theta(t)$.

Other choices of Γ are also investigated. For example, consider

$$
\Gamma : \overline{\psi} \to \psi : \begin{cases} \overline{\phi} = \phi_* + \overline{\psi} \\[2mm] \dot{\theta} = -\varepsilon\overline{\phi\phi^T}\theta \ , \quad \theta(0) = \theta_0 \\[2mm] \psi = -G(\overline{\phi^T}\theta) \end{cases}
\tag{4.6}
$$

Again solutions of (3.1)-(3.3) are fixed points of Γ and vice versa. In (4.6) Γ maps regressor error, i.e., deviations about ϕ_*. Application of the CMP to Γ in (4.6) yields conditions on transient behavior. These are verified by simulations.

4.2 Method of Averaging

One further possibility to predict transient behavior is to utilize the *method of averaging over finite time*. This analysis is not then dealing with stability. In this analysis we can return to the system of (2.1), (2.4), (2.5) which in state form is described by

$$
\dot{\theta} = \varepsilon\phi e \qquad , \qquad \theta(0) = \theta_0 \in \mathbb{R}^p
$$

$$
e = c^T x \qquad , \qquad \phi = Dx
\tag{4.7}
$$

$$
\dot{x} = A(\theta)x + B(\theta)w(t) \qquad , \qquad x(0) = x_0 \in \mathbb{R}^n
$$

(Recall that $\theta(t)$ is the *actual* adaptive parameter, not the deviation from θ_*). To apply averaging requires a time-scale decomposition (see, e.g., Riedle and Kokotovic (1986), Anderson et al., (1986)). This is accomplished by introducing the frozen system state $\overline{x}(t, \theta)$, i.e., for each *fixed* $\theta \in \mathbb{R}^p$,

$$
\overline{x}(t, \theta) = e^{tA(\theta)}x_0 + \int_0^t e^{(t-\tau)A(\theta)}B(\theta)w(\tau)d\tau
\tag{4.8}
$$

and the state error

$$
z(t) = x(t) - x(t, \theta(t))
\tag{4.9}
$$

The result is that for sufficiently small time intervals -- not necessarily small ϵ -- the parameter transient behaves like the solution to the linear system

$$\frac{d}{dt} (\theta - \theta_0) = \epsilon[\bar{\phi}(t, \theta_0)\bar{e}(t, \theta_0) - \bar{R}(t, \theta_0)(\theta - \theta_0)] \tag{4.10}$$

where $\bar{\phi}(t, \theta_0)$ and $\bar{e}(t, \theta_0)$ are the frozen system regressor and error, respectively, and $\bar{R}(t, \theta_0)$ is given by

$$\bar{R}(t, \theta_0) = \bar{\phi}(H_{ev}\bar{\phi})^T + \bar{e}(H_{\phi v}\bar{\phi})^T \tag{4.11}$$

In (4.11), H_{ev} and $H_{\phi v}$ have their usual definitions, vis a vis (2.9), except now are dependent on θ_0 rather than θ_*. Averaging analysis proceeds from here in the usual way and will be documented in the full version of the paper.

5. CONCLUSIONS

In this investigation of the transient properties of adaptive control systems we have shown that some of the interesting phenomena can be analyzed. The tools for analysis involve a combination of small gain theory, passivity, and the method of averaging with these all linked together by the Contraction Mapping Principal. Although each of these tools, in principal, involves straightforward calculations, it is clear that the level of complexity of a realistic adaptive system is well beyond hand calculation. Hence, an area for further work is in the development of software tools which can eliminate some of the tedious parts of the analysis.

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FIXED-POINT THEOREMS FOR STABILITY ANALYSIS OF ADAPTIVE SYSTEMS

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ABSTRACT
The use of fixed point theorems is considered for the stability analysis of adaptive systems. The particular fixed point theorems considered are the Contraction Mapping Principle of Banach and the Schauder Fixed Point Theorem. It is shown how the contraction property can be achieved by exponential stability of the homogeneous part of the linearized adaptive system. The region of linearization validity is estimated by considering fixed as well as adaptive tuned systems. Fixed-point theorems are shown also to be useful for a transient analysis of the adaptive system.

1. INTRODUCTION

A general structure for an adaptive system is shown in Fig.1 and is described by the operator equations

$$\begin{vmatrix} e \\ \phi \end{vmatrix} = H(\hat{\theta})w = \begin{vmatrix} H_{ew}(\hat{\theta}) \\ H_{\phi w}(\hat{\theta}) \end{vmatrix} w \qquad (1.1a)$$

$$\hat{\theta} = \Omega(e, \phi), \quad \hat{\theta}(0) = \hat{\theta}_0 \qquad (1.1b)$$

The adaptive system shown consists of three basic subsystems:

1. The **adaptation subsystem** Ω which uses the **error** signal e, the **regressor** signal ϕ, and the initial parameter value θ_0 to generate the parameter $\hat{\theta}$.
2. The **error subsystem** $H_{ew}(\hat{\theta})$ which is parametrically dependent on $\hat{\theta}$ and takes the exogenous inputs w - consisting of reference, disturbance and noises - and produces the error signal e used for adaptive parameter adjustment.
3. The **regressor subsystem** $H_{\phi w}(\hat{\theta})$, also dependent on $\hat{\theta}$, maps w into the regressor signal ϕ. The regressor is derived from measured signals and usually is constructed so as to represent the states of the system model.

The decomposition of the adaptive system shown in (1.1) is *non-standard* in that the plant and/or controller is not explicitly visible. This structure is chosen to highlight signal properties and general operator characteristics rather than plant/controller structures and parametrizations. A similar decomposition can be found in Kosut and Anderson (1986) and Anderson et al. (1986).

In this paper we will show how the Banach Fixed Point Theorem (the Contraction Mapping Principle) can be applied to study the local stability properties of (1.1). To apply the theorem we utilize the techniques of linearization about a tuned system trajectory. In this paper the constant parameter tuned system concept introduced in Kosut and Friedlander (1985) is extended to the more general case where the tuned system can be an ideal adaptive system.

2. THE TUNED SYSTEM AND LINEARIZATION

In this section we develop an incremental version of (1.1) which couples the deviations of e, ϕ, $\hat{\theta}$ from **tuned** signals e_*, ϕ_*, θ_*. We first develop the incremental form and then discuss the meaning of the tuned signals. For brevity we introduce the notation

$$x = \begin{vmatrix} e \\ \phi \\ \hat{\theta} \end{vmatrix}, \quad x_* = \begin{vmatrix} e_* \\ \phi_* \\ \theta_* \end{vmatrix}, \quad \tilde{x} = \begin{vmatrix} \tilde{e} \\ \tilde{\phi} \\ \tilde{\theta} \end{vmatrix} = x - x_* \qquad (2.1)$$

let $Q(x)$ denote the operator

$$Q(x) = \begin{vmatrix} H_{ew}(\hat{\theta})w \\ H_{\phi w}(\hat{\theta})w \\ \Omega(\phi, e) \end{vmatrix} \qquad (2.2)$$

Thus, (1.1) is equivalent to the single expression

$$x = Q(x) \qquad (2.3)$$

Using (2.1) with (2.3) gives

$$\tilde{x} = Q(x_* + \tilde{x}) - x_* \qquad (2.4)$$

which may also be written as

$$\tilde{x} = \delta + F(\tilde{x}) \qquad (2.5)$$

where

$$\delta = Q(x_*) - x_* = \begin{vmatrix} \delta_e \\ \delta_\phi \\ \delta_\theta \end{vmatrix} = \begin{vmatrix} H_{ew}(\theta_*)w - e_* \\ H_{\phi w}(\theta_*)w - \phi_* \\ \Omega(e_*, \phi_*)w - \theta_* \end{vmatrix} \qquad (2.6)$$

and

$$F(x) = Q(x_* + \tilde{x}) - Q(x_*) = \begin{vmatrix} H_{ew}(\theta_* + \tilde{\theta})w - H_{ew}(\theta_*)w \\ H_{\phi w}(\theta_* + \tilde{\theta})w - H_{\phi w}(\theta_*)w \\ \Omega(e_* + \tilde{e}, \phi_* + \tilde{\phi}) - \Omega(e_*, \phi_*) \end{vmatrix} \qquad (2.7)$$

The abstract nonlinear operator representation (2.5) is completely equivalent to the adaptive system (1.1). System (2.5) is referred to as the **error system** version of (1.1) corresponding to tuned signals e_*, ϕ_*, θ_*. One of the reasons for working with the error system rather than the original system is that robust stability of the original system is more easily expressed with respect to the system behaviour relative to the ideal (tuned) behaviour. We will return to this point later in the paper.

We now establish a conceptual framework for linearization. Suppose that $F(\tilde{x})$ is locally Frechet differentiable with respect to $\tilde{x}$ for all $\tilde{x}$ in some neighborhood of $\tilde{x} = 0$, e.g. $\|\tilde{x}\| \le r$ where $\|\cdot\|$ is a norm on a Banach space B. Then, for all $\|\tilde{x}\| \le r$, we may act as if there were a linear operator L such that

$$L(\tilde{x}) = \left| \frac{\partial F(\tilde{x})}{\partial \tilde{x}} \right| (\tilde{x}); \quad \tilde{x} = 0$$

$$-\begin{vmatrix} 0 & 0 & (\partial H_{ew}/\partial\theta)_{\theta_*}w \\ 0 & 0 & (\partial H_{\phi w}/\partial\theta)_{\theta_*}w \\ (\partial\Omega/\partial e)_{e_*,\phi_*} & (\partial\Omega/\partial\phi)_{e_*,\phi_*} & 0 \end{vmatrix} \begin{vmatrix} \bar{e} \\ \bar{\phi} \\ \bar{\theta} \end{vmatrix}$$

$$(2.8)$$

Hence, (2.5) is equivalent to

$$\bar{x} = \delta + L(\bar{x}) + \Delta(\bar{x}) \tag{2.9a}$$

where

$$\Delta(\bar{x}) = F(\bar{x}) - L(\bar{x}) \tag{2.9b}$$

Assuming for the moment that $H(\beta)w$ and $\Omega(e,\phi)$ are sufficiently regular operators, then L defined in (2.8) will be a linear integral operator with a locally integrable kernal function. Under suitable conditions on the existence of solutions to (2.9), we can express (2.9) equivalently by

$$\bar{x} = \bar{x}_L + N(\bar{x}) \tag{2.10a}$$

where

$$\bar{x}_L = (I-L)^{-1}\delta \tag{2.10b}$$

$$N(\bar{x}) = (I-L)^{-1}\Delta(\bar{x}) \tag{2.10c}$$

The signal $\bar{x}_L$ is referred to as the <u>linearized error system response</u>. Intuitively, if $N(0) = 0$ and $||N(\bar{x}_L)||$ is sufficiently small, then $||\bar{x}||$ will be small. This would establish that the adaptive system behaves very nearly like the tuned system. This result can be rigorously established by appealing to the Banach Fixed Point theorem, or as it is often referred to, the Contraction Mapping Principle. Before it is stated some definitions are needed.

If M is a subset of a Banach space B, and T is an operator mapping M→B, then T is a <u>contraction on</u> M if there is a constant $\sigma \in [0,1)$ such that

$$||Tx-Ty|| \leq \sigma||x-y||, \quad \forall x,y \in M \tag{2.11}$$

The constant σ is called the <u>contraction constant</u> for T on M. A <u>fixed point</u> of $\overline{T:M\to M}$ is a point $x\epsilon M$ such that $x = Tx$, i.e. $x\epsilon M$ is invariant under T.

Theorem 1: Contraction Mapping Principle
If M is a closed subset of a Banach space B and T:M→M is a contration on M, then T has a fixed point in M.   □

This statement of the Contraction Mapping Principle can be found in Hale (1980). Applying it to the error system (2.9) gives the following result.

Theorem 2: Linearization
Let B denote a Banach space and let B_r denote the closed subset of B defined by

$$B_r = \{f \in B : ||f|| \leq r\} \tag{2.12}$$

where r is a positive constant. Assume that:

A1: $N(0) = 0$ (2.13)

A2: $||N(x)-N(y)|| \leq K_N r||x-y||, \forall x,y\epsilon B_r$ (2.14)

Under these conditions, if $r>0$ satisfying

(1) $K_N r < 1$ (2.15)

(11) $||\bar{x}_L|| \leq r(1-K_N r)$ (2.16)

then a unique soluton $\bar{x}$ of (2.9) such that

$$||\bar{x}|| \leq r \tag{2.17}$$

Conditions A1 and A2 establish the local Lipschitz continuity of N(x). Moreover, $||x|| \leq r \Rightarrow ||N(x)|| \leq K_N r^2$. This quadratic bound on N(x) is insured if F(x) is Frechet differentiable for $||x|| \leq r$, because then it follows that $||x||_T \leq r \Rightarrow ||\Delta(x)||_T = O(||x||_T^2)$. The truncated norms are removed under the action of $(I-L)^{-1}$ on $\Delta(x)$, i.e. $N(x) = (I-L)^{-1}\Delta(x)$.

Proof: Let T be defined by the right hand side of $\overline{(2.10)}$, i.e.

$$T(x) = x_L + (I-L)^{-1}\Delta(x)$$

Observe that from (2.16), $T:B_r\to B_r$, i.e.

$$||Tx|| \leq ||x_L|| + K_N r^2 \leq r, \quad \forall x\epsilon B_r$$

Furthermore, (2.16) implies that T is a contraction on B_r, i.e.

$$||Tx-Ty|| \leq K_N r ||x-y||, \quad \forall x,y\epsilon B_r$$

with contraction constant $K_N r < 1$. By Theorem 1, there is a unique fixed point $x\epsilon B_r$ of T. Since $x = Tx$ is equivalent to (2.10) which is equivalent to (2.9), the theorem is proved.   □

Discussion
Theorem 2 asserts that the adaptive system signal trajectory is close to that of the the tuned system (r-small) if the linearized response $\bar{x}_L$ is small. Observe that the smallest value of r is limited by the size of $||\bar{x}_L||$ which depends on the interaction of $(I-L)^{-1}$ with δ. We will see later that δ varies considerably with the choice of tuned signals. Also, depending on this choice, both L and δ may contain integrators and thus, although $\bar{x}_L$ can be bounded, neither $(I-L)^{-1}$ nor δ may be individually bounded.

Theorem 2 gives qualitative information regarding the effect of the nonlinear term $N(\bar{x})$. Condition (2.17) shows that the stronger the nonlinearity (larger K_N), the smaller the allowable $||\bar{x}_L||$.

Beyond this type of information, Theorem 2 is a little opaque because the internal dynamics of the operator is not visible. Even so, we can still regard as good the intuitive idea that a small linearized error response is essential for robust behaviour near a tuned trajectory.

Before proceeding we remark on the possibility of relaxing the conditions of Theorem 2. This can be done by eliminating the uniqueness requirement and only establishing existence. One approach is to use the fixed point theorem of Schauder. The following theorem statement is in Hale (1980).

Theorem 3: Schauder Fixed Point Theorem
If M is a convex, compact subset of a Banach space B and T:M→M is continuous, then T has a fixed point in M.

The cost of eliminating uniqueness (T contractive on M) is that M must now be a compact, convex subset of B, whereas in Theorem 1, M need only be a closed subset of B. For example, suppose B = C[0,T], the Banach space of continuous, bounded functions on [0,T], $t<\infty$. Then a convex, compact

subset of M is the bounded and slew limited functions $f \in B$, i.e. $|f(t)| \leq r$, $\forall t \in [0,T]$ and $|f(t)-f(\tau)| \leq k|t-\tau|$, $\forall t, \tau \in [0,T]$. We point out in Section 4 that since T must be finite, Theorem 3 is useful for a transient analysis of the adaptive system.

For the moment, led by Theorem 1 and Theorem 2, we will examine the linearized response.

3. THE LINEARIZED RESPONSE

In this section we will examine the linearized system (2.10b), i.e.

$$\tilde{x}_L = \delta \cdot L(\tilde{x}_L) \qquad (3.1)$$

In order to be more specific about (3.1), it is necessary to be more specific about the structure of (1.1) and the choice of the tuned signals e_*, ϕ_*, θ_*.

3.1 Tuned System

In the formulation so far, the choice of tuned signals is arbitrary. The most common choice is where θ_* is a fixed parameter in R^p chosen as if the plant were known, and e_*, ϕ_* are generated as shown in Figure 2, i.e. from

$$\begin{vmatrix} e_* \\ \phi_* \end{vmatrix} = H(\theta_*)w = \begin{vmatrix} H_{ew}(\theta_*)w \\ H_{\phi w}(\theta_*)w \end{vmatrix} \qquad (3.2)$$

In this case we have from (2.6) that

$$\delta = \begin{vmatrix} 0 \\ 0 \\ \Omega(e_*, \phi_*) - \theta_* \end{vmatrix} \qquad (3.3)$$

Hence, the size of the linearized response is determined by the ability of the adaptation subsystem to hold θ near θ_*.

An alternative scheme for generating e_*, ϕ_*, θ_* is shown in Fig.3. In this case θ_* is not fixed but is the output of an adaptation subsystem. The difference between the system in Fig.3 and the system in Fig.1 is that $\bar{H}(\cdot)$ and $\bar{w}$ are ideal versions, or simplified models, of $H(\cdot)$ and w in Fig.1. For example, Fig.3 could represent an ideal adaptive system which is globally stable, whereas Fig.1 contains unmodelled dynamics and disturbances which remain unaccounted for in the ideal case. In this case we have

$$\delta = \begin{vmatrix} H_{ew}(\theta_*)w - \bar{H}_{ew}(\theta_*)\bar{w} \\ H_{\phi w}(\theta_*)w - \bar{H}_{\phi w}(\theta_*)\bar{w} \\ 0 \end{vmatrix} \qquad (3.4)$$

One of the major differences between the δ's in (3.3) and (3.4) is that δ in (3.4) is bounded wheres δ in (3.3) may not be bounded. The reason is that Ω typically contains integrators. However, in both cases $(I-L)^{-1}\delta$ is bounded, again, by the action of the integrators in L, i.e. the terms involving $\partial\Omega/\partial\phi$.

3.2 Adaptive System Structure

Consider the adaptive system of Fig. 4, described by

$$\begin{vmatrix} e \\ \phi \end{vmatrix} = G \begin{vmatrix} w \\ -v \end{vmatrix} = \begin{vmatrix} G_{ew} & G_{ev} \\ G_{\phi w} & G_{\phi v} \end{vmatrix} \begin{vmatrix} w \\ -v \end{vmatrix} \qquad (3.5a)$$

$$v = \phi'\theta \qquad (3.5b)$$

with adaptation subsystem given by the simple gradient algorithm

$$\Omega: \dot{\theta} = c\phi e, \quad \theta(0) = \theta_* \qquad (3.5c)$$

The operator G in (3.5a) is the "plant" and is assumed here to be a matrix of linear time-invariant operators with transfer function $G(s)$. Using the definition $\tilde{\theta} = \theta - \theta_*$ from (2.1),

system (3.5) is easily seen to be equivalent to

$$\dot{\tilde{\theta}} = c\phi e - \dot{\theta}_*, \quad \tilde{\theta}(0) = \theta_* - \theta_*$$

$$e = H_{ew}(\theta_*)w - H_{ev}(\theta_*)v$$

$$\phi = H_{\phi w}(\theta_*)w - H_{\phi v}(\theta_*)v$$

$$v = \phi'\tilde{\theta} - v - \phi'\theta_* \qquad (3.6)$$

where

$$H_{ew}(\theta_*) = G_{ew} - G_{ev}\theta_*'(I+G_{\phi v}\theta_*')^{-1}G_{\phi w}$$

$$H_{ev}(\theta_*) = G_{ev} - G_{ev}\theta_*'(I+G_{\phi v}\theta_*')G_{\phi v}$$

$$H_{\phi w}(\theta_*) = (I+G_{\phi v}\theta_*')^{-1}G_{\phi w}$$

$$H_{\phi v}(\theta_*) = (I+G_{\phi v}\theta_*')^{-1}G_{\phi v} \qquad (3.7)$$

In the case when e^*, ϕ^*, θ^* arise from the tuned system in Fig.2, where $\theta_* \in R^p$, then (3.6) becomes

$$\dot{\tilde{\theta}} = c\phi e, \quad \tilde{\theta}(0) = \theta_* - \theta_*$$

$$e = e_* - H_{ev}(\theta_*)v$$

$$\phi = \phi_* - H_{\phi v}(\phi'\tilde{\theta})v$$

$$v = c\phi'\tilde{\theta} \qquad (3.8)$$

The linearized system associated with (3.8) is defined by (2.10), (2.8) is:

$$\dot{\tilde{\theta}}_L = c[\phi_* e_* + \phi_* \tilde{e}_L + e_* \tilde{\phi}_L], \quad \tilde{\theta}_L(0) = \tilde{\theta}_*$$

$$\tilde{e}_L = -H_{ev}(\theta_*)(\phi_*'\tilde{\theta}_L)$$

$$\tilde{\phi}_L = -H_{\phi v}(\theta_*)(\phi_*'\tilde{\theta}_L) \qquad (3.9)$$

Observe that the "inputs" to (3.9) are ϕ_*, e_* and $\tilde{\theta}_*$. Since e_* is small by definition, the main cause of a large linearized response is the initial parameter error $\tilde{\theta}_* = \theta_* - \theta_*$. Since $H_{ev}(\theta_*)$ and $H_{\phi v}(\theta_*)$ are stable and e_* is small, it follows that the stability of (3.9) necessitates the stability of

$$\dot{\tilde{\theta}}_L = -c\phi_* H_{ev}(\theta_*)(\phi_*'\tilde{\theta}_L) \qquad (3.10)$$

This system, which is a linear time-varying system, can be analyzed by passivity methods [Kosut and Freidlander (1985)], small gain theory [Anderson et al. (1984)], and averaging [Riedle and Kokotovic (1985)]. When $H_{ev}(\theta_*)$ is not strictly passive, which is the normal case for actual systems, the latter two approaches offer similar stability results, namely, if $c < 0$ is sufficiently small, and if

$$\lambda_{min}\{avg[\phi_* H_{ev}(\theta_*)\phi_*']\} > 0 \qquad (3.11)$$

then (3.10) is exponentially stable, i.e., $M \geq 1$, $\alpha > 0$ such that solutions of (3.10) satisfy

$$|\tilde{\theta}_L(t)| \leq M e^{-\alpha(t-\tau)}|\tilde{\theta}_L(\tau)|, \quad \forall t-\tau \geq 0 \qquad (3.12)$$

If condition (3.11) fails to hold because at least one eigenvalue is negative, then (3.10) is unstable. This means that the adaptive system is locally unstable.

When $c>0$ is small, the adaptation is slow and (3.9) is a two-time scale system with $\tilde{\theta}_L$ the slow variable. Notice that if (3.11) holds, and hence (3.10) is exponentially stable, it is possible to achieve bounded $\tilde{\theta}_L, \tilde{e}_L, \tilde{\phi}_L$ for any initial parametrization $\tilde{\theta}_*$. Thus, any initial parametrization $\tilde{\theta}_*$ will be tolerated by the linearized system. In the actual system, limitations arise from the coupling between the linearized system and the neglected nonlinear terms as is evidenced by the restrictions on $|||\tilde{x}_L|||$

implied by (2.16). These can be partially relaxed by initially taking into account the two-time scale system behaviour and developing a two-time scale linearization theorem analogous to Theorem 2. Such results, derived from the method of averaging of Bogolyubov and Mitropolski (1955), have been developed for adaptive systems, e.g. Astrom (1984), Reidle and Kokotovic (1985b), Bodson et al (1985), Anderson et al (1986). Qualitatively, these results assert that if θ_* is a stable parametrization, then for sufficiently small $\varepsilon>0$, $\theta(t)$ will move slowly through the constant parameter stability set - provided (3.11) holds - and arrive at an $O(\|e_*\|_\infty)$ neighborhood of a tuned setting θ_*. Moreover, θ_* can be determined from avg$[e_*\phi_*] = 0$ provided $e_*\phi_*$ has an average value.

Note: In general, the exponential stability of the linearized system will insure the contraction property required in Theorem 2. How to ensure an exponential stability in the linearized system is not completely solved, but at present it can be ensured from passivity, small gain, or averaging analyses, see for example Anderson et al (1986). The interesting aspect of the contraction (fixed-point) argument is that the mechanism for the contraction need not be specified.

In the case when e_*,ϕ_*,θ_* arise from the tuned system in Fig. 3, then (3.6) can be written as

$$e = e_* + \delta_e - H_{ev}(\theta_*)v$$

$$\phi = \phi_* + \delta_\phi - H_{\phi v}(\theta_*)v$$

$$v = \phi'\tilde\theta$$

$$\dot{\tilde\theta} = \varepsilon[\phi e - \phi_* e_*] , \quad \tilde\theta(0) = 0, \tag{3.13}$$

where δ_e,δ_ϕ are given by (3.4), i.e.

$$\delta_e = H_{ew}(\theta_*)w - \bar H_{ew}(\theta_*)\bar w$$

$$\delta_\phi = H_{\phi w}(\theta_*)w - \bar H_{\phi w}(\theta_*)\bar w \tag{3.14}$$

The linearized systems associated with (3.13) as defined by (2.10), (2.8) is:

$$\bar e_L = \delta_e - H_{ev}(\theta_*)(\phi_*\tilde\theta_L)$$

$$\bar\phi_L = \delta_\phi - H_{\phi v}(\theta_*)(\phi_*\tilde\theta_L)$$

$$\dot{\tilde\theta}_L = \varepsilon[\phi_*\bar e_L + e_*\phi_L], \quad \tilde\theta_L(\cdot) = 0 \tag{3.15}$$

Comparing (3.15) with (3.9), the operators $H_{ev}(\theta_*)$, $H_{\phi v}(\theta_*)$ in (3.15) are linear and time-varying, because θ_* is time-varying (Fig.3), whereas in (3.9) the operators are linear time-invariant because θ_* is fixed (Fig.2). Another difference is that the "inputs" in (3.15) are usually significantly smaller than in (3.9). In (3.15) the inputs are δ_e,δ_ϕ from (3.14), whereas in (3.9) the inputs are $\phi_* e_*$ and δ_*. Certainly $\phi_* e_*$ is small because e_* is small. Also, δ_e,δ_ϕ can be small if the ideal system (Fig.3) is not significantly different from the actual system (Fig.2). However, the initial parameter error $\tilde\theta_*$ can be quite large and, as already discussed, can limit the region of linearization validity, e.g. (2.16), whereas in (3.15) the effect of $\tilde\theta_*$ as an input is subsumed by the linearization about the tuned trajectory of the ideal adaptive system (Fig.3).

The stability analysis of (3.15) proceeds in much the same way as the analysis to (3.9). Two phases of the analysis can be distinguished. There is first a transient phase, during which θ_* is not necessarily moving slowly even if ε is small. For example, when the initial parametrization is near instability it is quite possible that the behaviour of θ_* is erratic - such behaviour has been seen in simulations. After the transient phase, θ_* may

then move slowly and begin to approach a constant tuned setting. Let us first consider the terminal phase, after the transients have disappeared.

Suppose that as $t\to\infty$, $\theta_*(t)\to\bar\theta_*\in HP$, where $\bar\theta_*$ is like the tuned parametrization in Fig.2. In the ideal case we may have $e_*(t)\to0$ and $\phi_*(t)\to\bar\phi_*(t)$ exponentially fast, where $\bar\phi_*(t)$ is like $\phi_*(t)$ in Fig.2. Thus, neglecting the effect of the exponentially decaying terms, (3.15) becomes

$$\dot{\tilde\theta}_L = \varepsilon\bar\phi_*\bar e_L , \quad \tilde\theta_L(0) = 0 \tag{3.16}$$

$$\bar e_L = \bar\delta_e - H_{ev}(\bar\theta_*)(\bar\phi_*\tilde\theta_L)$$

$$\bar\phi_L = \bar\delta_\phi - H_{\phi v}(\theta\theta_*)(\bar\phi_*\tilde\theta_L)$$

where

$$\bar\delta_e = H_{ew}(\bar\theta_*)w$$

$$\bar\delta_\phi = H_{\phi w}(\bar\theta_*)w - \bar H_{\phi w}(\bar\theta_*)\bar w \tag{3.17}$$

Since $H_{ev}(\bar\theta_*)$ and $H_{\phi v}(\bar\theta_*)$ are stable LTI operators, the stability of (3.16) and (3.15) depends on the stability of

$$\dot{\tilde\theta}_L = -\varepsilon\bar\phi_* H_{ev}(\bar\theta_*)(\bar\phi_*\tilde\theta_L) \tag{3.18}$$

which is precisely the system in (3.10), and so the methods of analysis discussed after (3.10) also apply. To reiterate, the main difference between (3.15) and (3.9) is the input magnitude. Hence, $\|\tilde x_L\|$ is much smaller in (3.15) than in (3.9), and so the limitation on linearization validity as expressed in (2.16) is more easily satisfied.

4. TRANSIENT ANALYSIS

The drawback of carrying out the analysis of the error system described by (3.13) is that the properties of the error signals $\bar e,\bar\phi,\tilde\theta$ depend on the properties of e_*,ϕ_*,θ_* as generated from the ideal adaptive system in Fig.3. Although a global analysis of such systems is available [see, e.g. Narendra, Lin and Valavani (1980), Kosut and Friedlander (1985)] the results are qualitative and the signal bounds are coarse. Thus a detailed description of the ideal system behaviour is not available. Since the averaging analysis referenced in Section 3 can describe the system behaviour in some detail for small $\varepsilon>0$ with stable initial parametrizations, it follows that a transient analysis of the ideal system (Fig.3) is needed for other initial parametrizations. This includes both stable and unstable initial parameterizations. At the same time we need a transient analysis when ε is not necessarily small, or as small as required by averaging theory, or when ε switches from large to small values, e.g. as in recursive least squares with a constant forgetting factor. In this latter case the adaptation mechanism can be written as in Ljung and Soderstrom (1983),

$$\dot\theta = \varepsilon R^{-1}\phi e \qquad \theta(\cdot) = \theta_*$$

$$\dot R = \varepsilon(\phi\phi' - R), \quad R(\cdot) = R(\cdot)' > 0 \tag{4.1}$$

Here ε is small but the choice of $R(0) = (\varepsilon/\sigma)1$ with large $\sigma > 0$ results in an effective large initial gain $\sigma 1$. Other schemes can also be envisioned when ε in (3.5c) changes size depending on the size of $e(t)$ on some measure, e.g. $\|e\|$ over some time window $[t-T,t]$, etc.

The Contraction Mapping Principle (Theorem 1) can, in principle, be applied to the transient analysis problem. Suppose that the ideal system (Fig.3) is given by

$$\dot{\tilde\theta}_* = -\varepsilon\phi_*\phi_*'\tilde\theta_* , \quad \tilde\theta_*(0) = \theta(0) - \bar\theta_*$$

$$\phi_* = \bar\phi_* - H_\phi(\bar\theta_*)(\phi_*\tilde\theta_*)$$

$$e_* = 0 \tag{4.2}$$

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The notation is a bit awkward, but $\tilde{\theta}_* = \theta_* - \bar{\theta}_*$ where θ_* is the tuned time-varying estimate in Fig.3; $\bar{\theta}_*$ is the fixed final value for which $e_* = 0$; and ϕ_* is the final regressor function when $\theta_* = \bar{\theta}_*$. In order to utilize Theorem 1 to establish the transient behaviour of (4.2) it is necessary to identify the operator T and the closed space M. For example, one choice is to define T as the mapping $\psi \to \theta$ defined implicitly by

$$\dot{\theta} = -\varepsilon\phi\phi^*\theta, \quad \theta(0) = \theta(0) - \bar{\theta}_*$$

$$\phi = \bar{\phi}_* - H_{\phi\psi}(\bar{\theta}_*)(\phi^*\psi) \tag{4.3}$$

The set M can be chosen as

$$M = \{\theta \varepsilon C[0,T]: \|\theta - \theta_B\| \le \eta, \; \theta_B(t) = e^{tB}\theta(0)\} \tag{4.4a}$$

with norm

$$\|r\|_M = \sup_{t\varepsilon[0,T]} |r(t)| \tag{4.4b}$$

where $B\varepsilon R^{p \times p}$ with $Re\lambda(B) < 0$. The relationships between B,T, and η are critical in the transient and will be discussed in the presentation as space in the paper is limited. Note also that the transient analysis can be carried out by appealing to the Schauder Fixed Point Theorem (Theorem 3) where the space M above is modified so that θ is slew limited, i.e. $|\theta(t) - \theta(\tau)| \le k|t-\tau|$, $\forall t,\tau\varepsilon[0,T]$. Then M is a compact, convex subset of $C[0,T]$ as discussed after the statement of Theorem 3. In this latter case we are only able to establish existence. However, if the map T is locally contractive on $[0,T]$, then uniqueness is also established.

CONCLUSIONS

Two fixed point theorem (FPT) on Banach spaces have been presented alongside an operator formulation of adaptive control in terms of signals θ,ϕ,e - collectively denoted x. The Banach spaces under consideration are various function spaces $\{x(t):t\varepsilon T\}$ for T being the time-index set and compact subsets in this space correspond to collections of neighbouring (under the appropriate norm) time functions. A fixed point of the operator equations of adaptive control is identified with the complete time history of $x(=\phi,\theta,e)$ for the adaptive control problem. The role of the FPT's is to allow derivation of conditions for these signals to remain close as functions of time to nominal, well-behaved trajectories ϕ_*,θ_*,e_*.

The formalism of appealing to these FPT's dictates that a formulation such as (2.5) is achieved for an error system. Our approach to this is to involve a linearization about our nominal values. Local contractivity of the nonlinear operator equation is implied by exponential stability of the linearized adaptive control problem. This implication is at the very heart of our method. Smallness of the additive operators due to unmodeled dynamics, linearization, etc., is then invoked to prove the good behaviour of the adaptive control system via the FPT.

It is clear that there are many aspects affecting the quantitive application of these ideas - should anyone ever deem this appropriate - and

foremost among these is the choice of tuning signals. Two particular natural choices for signals are advanced which have respective advantages. Other effects are such things as sensitivity of the controlled plant to parameter variations about θ_* (connected with the validity of linearization), actual magnitude chosen for the gain ε (reflecting the trade off between contraction constant/exponential degree of stability and perturbation magnitude), choice of reference signals, etc., which reflect the influence of the actual plant, specified control objective and law, and specific adaptation rule on the behaviour of the complete adaptive system. Our thesis is that the analysis, via e.g. FPT's, of the signal-based operator formulation depicted in Figure 1 is a particularly natural and flexible approach to the analysis of robust adaptive control. This technique provides considerable insight into the qualitative issues implicated in achieving robust adaptive control. The quantitative issues require very much more specific site information (Poubell et al.).

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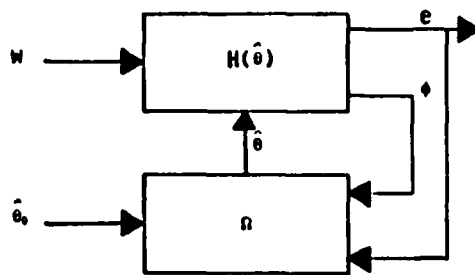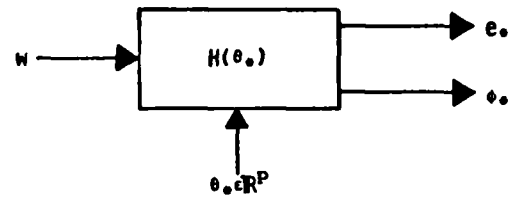
Fig.1: The Adaptive System

Fig.2: Tuned System - Fixed θ_o

Fig.3: Tuned System - The Ideal Adaptive System

Fig.4: Adaptive System

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HOW EXCITING CAN A SIGNAL REALLY BE?

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ABSTRACT

The rate of parameter convergence in a number of adaptive estimation schemes is related to the smallest eigenvalue of the average information matrix determined by the regression vector. Using a very simple example, we illustrate that the input signals that maximize this minimum eigenvalue may be quite different from the input signals that optimize more classical input design criteria, e.g. D-optimal criterion.

Key words: Exponential convergence, persistence of excitation, experiment design

1. PREAMBLE

Listen: The concept of "persistently exciting" (PE) signals has invaded the adaptive systems literature at an exponential rate. Currently, many papers on adaptive estimation or adaptive control contain long derivations proving that there exists some $T>0$, some $t_0 \geqslant 0$, and some $\alpha>0$, $\beta>0$ such that a certain regression vector $\phi(t)$ satisfies the following condition

$$\beta I \geqslant \frac{1}{T} \int_{t}^{t+T} \phi(\tau)\phi^{T}(\tau) \ d\tau \geqslant \alpha I \quad \text{for all } t \geqslant t_0. \tag{1}$$

This is the celebrated persistency of excitation condition. The regression vector $\phi(t)$ can take many forms, depending on the problem, but the following form is typical:

$$\phi^{T}(t) = \frac{1}{(s+\gamma)^{n-1}} [u(t) \ \dot{u}(t) \ldots u^{(n-1)}(t) \ y(t) \ \dot{y}(t) \ldots y^{(n-1)}(t)] \tag{2}$$

We have assumed a single input single output (SISO) system for simplicity, with input $u(t)$ and output $y(t)$; γ is a positve constant and n is the order of the system.

It is beyond the reach of this short technical note to dwell on the many occurrences of the persistency of excitation condition in adaptive estimation and adaptive control theory, but for those readers unfamiliar with this field let us just say that this condition is often appealed to to establish the exponential convergence of a linear time-varying error system. This insures the exponential convergence of all internal variables to their desired values in the idealized case (constant system, exact model matching, etc.) and their boundedness in certain non-ideal cases (time-varying parameters, unmodelled dynamics, etc). And so it goes.

The simplest and most informative occurrence of the PE condition is in the analysis of gradient algorithms for the estimation of a parameter vector. The error equations have the form

$$\dot{e}(t) = c\phi(t)e(t) \tag{3}$$

where $\theta(t)$ is the parameter estimation error, $c>0$ is the adaptation gain, $\phi(t)$ is the regression vector and $e(t)=-\phi^T(t)\theta(t)$ is an error signal. It can be shown that, subject to ϕ satisfying (1), (3) is uniformly exponentially convergent to zero. Further, if additionally c is small (actually $c\beta T<<1$) then the convergence rate of (3) is bounded below by

$$k = c \, \lambda_{min}(R) + O(c^2)$$

where

$$R = \lim_{T\to\infty} \frac{1}{T} \int_0^T \phi(\tau)\phi^T(\tau)d\tau$$

assuming this limit exists. That is

$$|\theta(t+\tau)| < Ke^{-k\tau}|\theta(t)|$$

for all t and τ and some $1<K<\infty$ fixed, with k being approximately linear in c [1,2,3].

One is then drawn to ask how to maximize the convergence rate of the error system (3) by manipulating designer variables – specifically c and ϕ. In most adaptive situations the algorithm gain c is constrained to be small relative to the regressor magnitude by the requirements of noise rejection – the variance of the parameter error in adaptive filtering is typically proportional to $c\beta^{1/2}$ [4] – so that the small c assumption concurs with engineering dictates. The meaningful subproblem then is: given that c is already small, how can we best choose ϕ to achieve maximum convergence rate or, more fully, with ϕ determined by (2) how should we choose $u(t)$?

Our aim in this paper is not to develop broad new frontiers in the robustness of adaptive systems, on which entire books could be written [1], but rather to analyse critically the PE condition itself with a view to answering some of the practical questions raised above. The overwhelming body of work so far has been algebraic in nature in establishing conditions for the regression vector ϕ of some particular adaptive system to satisfy (1) for some α and β. Our discussion above shows that it makes good sense to keep $c\beta$ (or $c\beta^{1/2}$) small, and that, if $c\beta T$ is sufficiently small, the

convergence rate is proportional to $c\alpha$. It follows that it is desirable to generate regressors $\phi(t)$ that will maximize α and α/β. The question we want to investigate is: "How can we achieve this by a proper choice of $u(t)$?".

This is clearly an optimal input design-type question. Input design was more fashionable a decade ago in the system identification literature, where persistency of excitation also originated. However, most of the effort was aimed at maximizing the determinant of the information matrix (this is called D-optimality), rather than its minimum eigenvalue, or the inverse of its condition number. In this note we examine the very simple case:

$$\phi(t) = \begin{vmatrix} u(t) \\ y(t) \end{vmatrix} \quad , \quad H(s) = \frac{b}{s+a} \quad , \quad y(t) = H(s)u(t)$$

and we solve the optimal input design problem for three different criteria. We seek the input $u(t)$ that maximizes, respectively, $\lambda_{min}(R)$, $\lambda_{min}(R)/\lambda_{max}(R)$, and for comparison purposes, det R. Given the connections we have established with the convergence rate of an adaptive algorithm, we shall show that maximizing the determinant leads to a rather poor input design. The main reason for its popular use is probably the simplicity of computation of the optimal input.

Some of these issues of experiment design in an adaptive systems context have been raised before [5] simply to emphasize the connection. Here we stress the unexpected difference between adaptive experiment design and optimal off-line experiment design.

One reason for our interest in this question arises from experimental attempts to generate PE signals for simple linear systems, with an adequate "richness" of the regression vector $\phi(t)$ leading to a particular minimum convergence rate of the error variables. It is often thought that only an academic researcher with a very twisted mind could generate signals that will violate the PE condition (e.g. try $u(t) = \cos t^2$ going through a low

pass filter). As it turns out, the problem is not to generate a $\phi(t)$ that satisfies (1) for some $\alpha>0$, but to obtain an α that is large enough to produce a reasonable convergence rate for the adaptive algorithm. In other words: How does one turn exponentially slow convergence into exponentially fast convergence? Finally, we wish to mention that we are by no means the first to discover that exponential convergence can be exceedingly slow, and some authors have conjectured that the slow convergence was probably due to a poor choice of input signal (see e.g. [6]). Given the practical importance of the question we raise, it is surprising that almost no attempts have been made to answer it. The purpose of this note is to give some very preliminary answers based on the analysis of the simplest possible case. We believe that our results provide a lot of insight, at least for us, which may help crack the more general case.

2. EXAMPLI GRATIA*

In this section we consider the situation of a two dimensional regression vector:

$$\phi(t) = (u(t) \ (Hu)(t))^T \ ; \quad t \in \mathbb{R}^+ \tag{2.1}$$

consisting of the input $u(t)$ and the filtered signal $(Hu)(t)$ – where H is a strictly stable, causal, linear time invariant operator with transfer function

$$H(s) = \frac{b}{s+a} \quad s \in \mathbb{C}, \ a>0 \tag{2.2}$$

This is typical for the adaptive identification or control of a first order plant. The design variable is the input $u(t)$, which we want to select so as to guarantee "optimal" performance of the adaptive system. Under the mild assumption that the input allows the definition of a power spectrum [1], this boils down to investigating the properties of the matrix:

$$R = \lim_{T \uparrow \infty} \frac{1}{T} \int_0^T \phi(t)\phi(t)^T dt \tag{2.3}$$

The input's power spectrum is defined as

$$S(\omega) = \int_{-\infty}^{+\infty} r(\tau)e^{j\omega\tau} d\tau; \quad \omega \in \mathbb{R} \tag{2.4}$$

where $r(\tau)$ is by assumption Fourier transformable, and is defined via

$$r(\tau) = \lim_{T \uparrow \infty} \frac{1}{T} \int_\alpha^{\alpha+T} u(t)u(t+\tau)dt; \quad \tau, \alpha \in \mathbb{R} \tag{2.5}$$

where the limit exists uniformly in α (for $u(t)$ defined on $\mathbb{R}^+$, $\alpha \geq \min(0,-\tau)$). Under mild conditions the spectrum uniquely determines the input.

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* e.g.

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Obviously (2.5) and (2.2) imply the existence of R (2.3), moreover R is given by:

$$
R = \begin{vmatrix} \int_{-\infty}^{+\infty} S(\omega)\,d\omega & \int_{-\infty}^{+\infty} S(\omega)\mathrm{Re}H(j\omega)\,d\omega \\ \int_{-\infty}^{+\infty} S(\omega)\mathrm{Re}H(j\omega)\,d\omega & \int_{-\infty}^{+\infty} S(\omega)|H(j\omega)|^2\,d\omega \end{vmatrix} \tag{2.6}
$$

where in this case

$$
\mathrm{Re}H(j\omega) = \frac{b}{a}\ \frac{1}{\frac{\omega^2}{a^2}+1} \tag{2.7}
$$

$$
|H(j\omega)|^2 = \frac{b^2}{a^2}\ \frac{1}{\frac{\omega^2}{a^2}+1} \tag{2.8}
$$

We compare the following three input selection criteria:

Selection Criteria:

Over the class of input functions ($u(t)$, $t\in R^+$), which have a power spectrum (as defined in (2.4)-(2.5)) and which satisfy the constraint:

$$
0 < \int_{-\infty}^{+\infty} S(\omega)\,d\omega < 1 \tag{2.9}
$$

maximise, either

(C1) $\det(R)$

or

(C2) $\lambda_{min}(R)$

or

(C3) $\lambda_{min}(R)/\lambda_{max}(R)$.

Solution:

Define

$$\alpha = \frac{\int_{-\infty}^{+\infty} S(\omega) \frac{1}{\frac{\omega^2}{a^2}+1}\,d\omega}{\int_{-\infty}^{+\infty} S(\omega)\,d\omega} \tag{2.10}$$

and

$$M(\alpha) = \begin{vmatrix} 1 & \alpha\frac{b}{a} \\ \alpha\frac{b}{a} & \alpha\frac{b^2}{a^2} \end{vmatrix} \tag{2.11}†$$

Surprisingly, we have that

$$R = M(\alpha) \int_{-\infty}^{+\infty} S(\omega)\,d\omega \tag{2.12}$$

and

$$0 < \alpha < 1 \tag{2.13}$$

Therefore, the optimal input functions acording to (C1) or (C2) satisfy (2.9) with equality, whilst for (C3) the magnitude of the total input power is immaterial. Consequently, the optimal inputs are characterized as:

C1-optimal: $\int_{-\infty}^{+\infty} S(\omega) \frac{1}{\frac{\omega^2}{a^2}+1}\,d\omega = \alpha_1^*; \quad \int_{-\infty}^{+\infty} S(\omega)\,d\omega = 1$ (2.14)

C2-optimal: $\int_{-\infty}^{+\infty} S(\omega) \frac{1}{\frac{\omega^2}{a^2}+1}\,d\omega = \alpha_2^*; \quad \int_{-\infty}^{+\infty} S(\omega)\,d\omega = 1$ (2.15)

C3-optimal: $\int_{-\infty}^{+\infty} S(\omega) \frac{1}{\frac{\omega^2}{a^2}+1}\,d\omega = \alpha_3^*\beta; \quad \int_{-\infty}^{+\infty} S(\omega)\,d\omega = \beta < 1$ (2.16)

where the $\alpha_i^* \in (0,1)$ maximise respectively $\det M(\alpha)$, $\lambda_{min}M(\alpha)$ and $\lambda_{min}M(\alpha)/\lambda_{max}M(\alpha)$ over $\alpha \in (0,1)$; and β is any number in $(0,1]$.

---

† This matrix M should not be confused with the M matrix of Poubelle et.al.[6].

After some simple calculations, we arrive at:

$$\alpha_1^* = 1/2 \tag{2.17}$$

$$\alpha_2^* = \frac{2}{4+\frac{b^2}{a^2}} \quad \in (0,\frac{1}{2}) \tag{2.18}$$

$$\alpha_3^* = \frac{1}{2\frac{b^2}{a^2}+1} \quad \in (0,1) \tag{2.19}$$

Equations (2.14)-(2.19) characterize all "optimal" solutions. In order to get some more insight, we verify whether there exist optimal inputs of the form

$$u(t) = \sqrt{2} \cos\omega^* t, \ t \in \mathbb{R}^+, \ \omega^* \in \mathbb{R}^+ \tag{2.20}$$

with power spectrum:

$$S(\omega) = \frac{1}{2}(\delta(\omega-\omega^*)+\delta(\omega+\omega^*)) \tag{2.21}$$

Solving for ω^* we find respectively for C1, C2 and C3:

$$\omega_1^* = a \tag{2.22}$$

$$\omega_2^* = a(1+\frac{b^2}{2a^2})^{\frac{1}{2}} \tag{2.23}$$

$$\omega_3^* = a(2\frac{b^2}{a^2})^{\frac{1}{2}} \tag{2.24}$$

For this type of input (2.20) we collected in Table 1 the relevant quantities (det R, $\lambda_{min}(R)$, $\lambda_{min}(R)/\lambda_{max}(R)$), as a function of b and a. In Table 2, the same quantities are displayed for $b/a = 1$.

For the purely sinusoidal input u(t) (2.20), the dependence of the determinant of R on the frequency ω is displayed in Figure 1. (Notice that the determinant is normalised by the D.C. gain squared.) The minimum eigenvalue and the condition number are displayed as functions of frequency respectively in Figure 2 and 3. The full line corresponds to a D.C. gain of 10, whilst the dotted line correponds to a D.C. gain of 1.

Table 1:

Criterion	ω^*	det R	$\lambda_{min}(R)$	$\dfrac{\lambda_{min}(R)}{\lambda_{max}(R)}$
C1	a	$\dfrac{1}{4}\dfrac{b^2}{a^2}$	$\frac{1}{2}((1+\dfrac{b^2}{2a^2})-\sqrt{1+\dfrac{1}{4}\dfrac{b^4}{a^4}})$	$\dfrac{1+\dfrac{b^2}{2a^2}-\sqrt{1+\dfrac{1}{4}\dfrac{b^4}{a^4}}}{1+\dfrac{b^2}{2a^2}+\sqrt{1+\dfrac{1}{4}\dfrac{b^4}{a^4}}}$
C2	$a\sqrt{1+\dfrac{b^2}{2a^2}}$	$\dfrac{2(2+\dfrac{b^2}{a^2})\dfrac{b^2}{a^2}}{(4+\dfrac{b^2}{a^2})^2}$	$\dfrac{\dfrac{b^2}{a^2}}{4+\dfrac{b^2}{a^2}}$	$\dfrac{4a^2}{b^2}+2$
C3	$a\sqrt{2\dfrac{b^2}{a^2}}$	$\dfrac{2\dfrac{b^4}{a^4}}{(2\dfrac{b^2}{a^2}+1)^2}$	$\dfrac{(3\dfrac{b^2}{a^2}+1)-\sqrt{\dfrac{b^4}{a^4}+6\dfrac{b^2}{a^2}+1}}{2(2\dfrac{b^2}{a^2}+1)}$	$\dfrac{3(\dfrac{b^2}{a^2}+1)-\sqrt{\dfrac{b^4}{a^4}+6\dfrac{b^2}{a^2}+1}}{(3\dfrac{b^2}{a^2}+1)+\sqrt{\dfrac{b^4}{a^4}+6\dfrac{b^2}{a^2}+1}}$

Table 2: (D.C. Gain = 1)

Criterion	ω^*	det R	$\lambda_{min}(R)$	$\dfrac{\lambda_{min}(R)}{\lambda_{max}(R)}$
C1	a	$\dfrac{1}{4}$	$\dfrac{3-\sqrt{5}}{4}$	$\dfrac{3-\sqrt{5}}{3+\sqrt{5}}$
C2	$a\sqrt{\dfrac{3}{2}}$	$\dfrac{6}{25}$	$\dfrac{1}{5}$	$\dfrac{1}{6}$
C3	$a\sqrt{2}$	$\dfrac{2}{9}$	$\dfrac{2-\sqrt{2}}{3}$	$\dfrac{2-\sqrt{2}}{2+\sqrt{2}}$

3. DISCUSSION

Consideration of input sequences that maximize the minimum eigenvalue of R (as in criterion (C2) of the preceding section) is encouraged in the introduction. This point is argued persuasively, and with more detail, in [5]. What is not examined in [5] is the difference in the subsequent input choice relative to the more common objective of determinant maximization [8]. As stated earlier, the purpose of this note is to draw attention to this difference via examination of a simple example.

1. The most immediate observation is that maximizing the minimum eigenvalue of the information matrix yields a different "optimal" input sequence from the one derived by maximizing the determinant or the ratio of the minimum and maximum eigenvalues.

2. In the first order example of Section 2, the frequency ω^* of the selected sinusoid is the breakpoint (or 3dB) frequency of the plant in (2.2) with determinant maximization; while ω^* is larger for minimum eigenvalue and minimum-to-maximum eigenvalue ratio maximization objectives. In fact, as the D.C. gain $(^b/_a)$ of the plant increases, so do the selected input frequencies for the minimum and minimum/maximum eigenvalue maximization criteria.

3. One interpretation of the tradeoff inherent with ω^* selection for minimum eigenvalue maximization is its tendency to make R in (2.12), or equivalently $M(\alpha)$ in (2.11), equal the identity matrix by attempting to keep the plant gain close to one while simultaneously attempting to achieve a 90° phase shift in order to null the off-diagonal terms on average. Table 2 indicates the compromise between these conflicting objectives when $^b/_a = 1$. This interpretation also explains why the input frequency that maximizes $\lambda_{min}(R)$ increases as the plant D.C. gain increases.

4. We should also note the nonuniqueness of the "optimal" u, unless, as in our example, the input power is constrained and (u) is assumed to be

composed of a number of sinusoids. We refer to [8, Chapter 6] for further comment.

5. One extension of "optimal" input selection would be to incorporate a measure of sensitivity to imprecisely known plant parameters. Such plant model imprecision is actually the motivation for identification procedures and the associated input selection. The example in Section 2 clearly indicates that the "optimal" input, by the various criteria, is a function of the "unknown" plant parameters.

6. We note that in this example the sensitivity of the λ_{min} design criterion is better than for the determinant criterion as is clear from Figures 1 and 2. Further, for all three criteria the penalty for using higher than optimal frequency appears less than that for using a lower frequency than optimal, and the sensitivity for the λ_{min} criterion in this example improves with increasing D.C. gain. This criterion is the one of prime interest for the convergence rate.

Finally, as discussed in [5], optimal input design questions are perhaps better posed in an adaptive estimation context than in an off-line identification situation. This is because, as the adaptive identifier learns more about the system, the input signals can be adjusted according to the relevant criterion. The insensitivity to imprecise knowledge of system parameters is then clearly advantageous and of relevant concern.

4. CONCLUSIONS

We have argued that, in the case of slow adaptation, the smallest eigenvalue and the condition number of the average information matrix determined by the regression vector should be considered as input design criteria in order to maximize the rate of exponential convergence. We have then performed this optimal input design in the simplest possible case, which allows a complete description of all optimal solutions. One should

be very careful in extrapolating the conclusions of this simple example to more general situations, but we believe that the main merit of our note is to draw attention to this problem because it follows from our analysis that:

1. The optimal inputs that result from our design criteria are quite different from those obtained using the classical D-optimality criterion for optimal input design in off-line parameter identification.

2. In some adaptive control schemes it is presently being advocated to concentrate the input signals in low frequency regimes in order that an average signal positivity condition (related to strict positive realness) is achieved. The results here indicate that this may cause an attendant decrease in the level of persistence of excitation.

Given the practical importance of optimizing the rate of parameter convergence, and given that this preliminary analysis points in a direction opposite from presently prevailing ideas, we believe that this problem deserves much more attention.

"Nyuk, nyuk, nyuk" - Curly Howard

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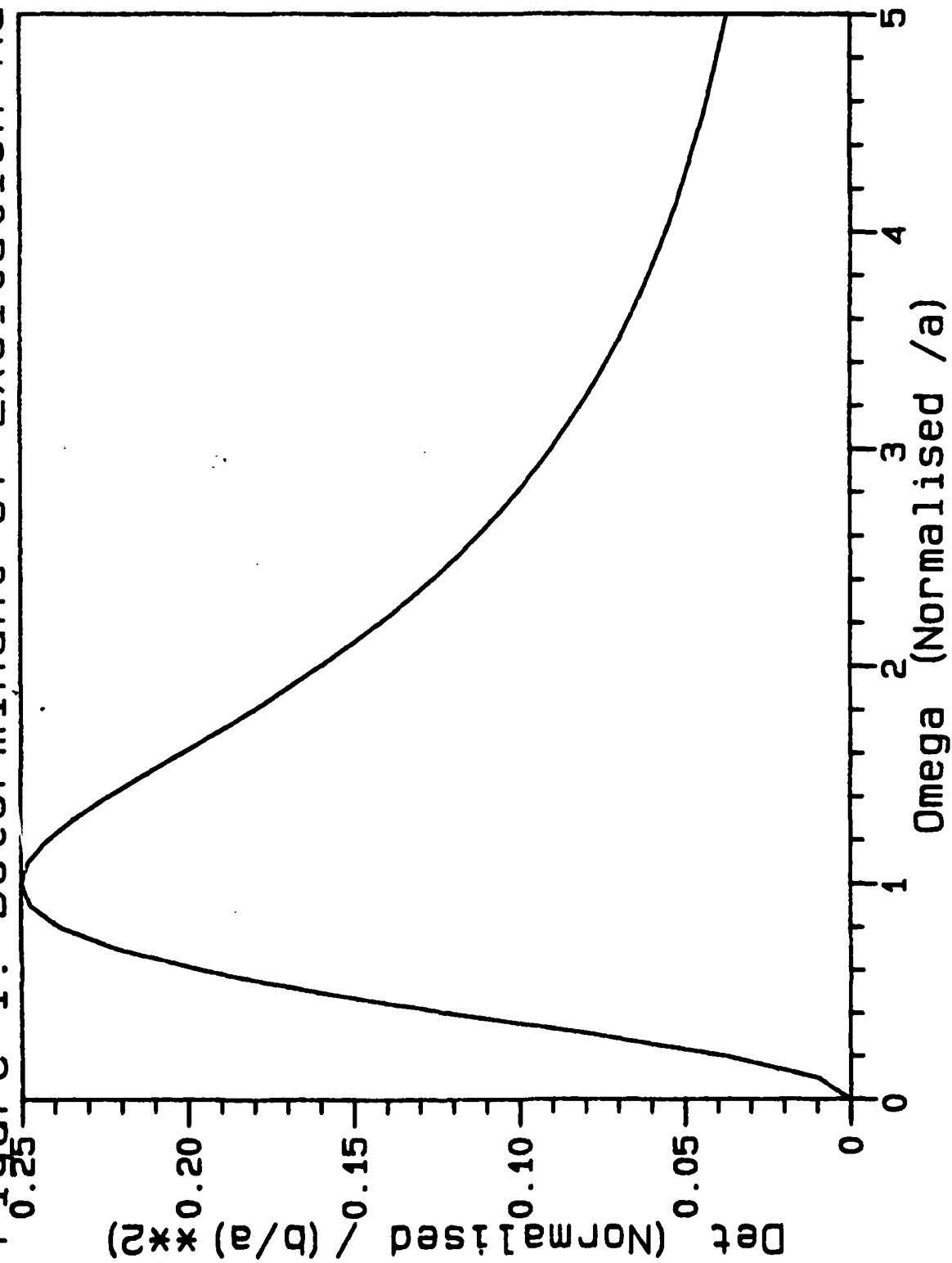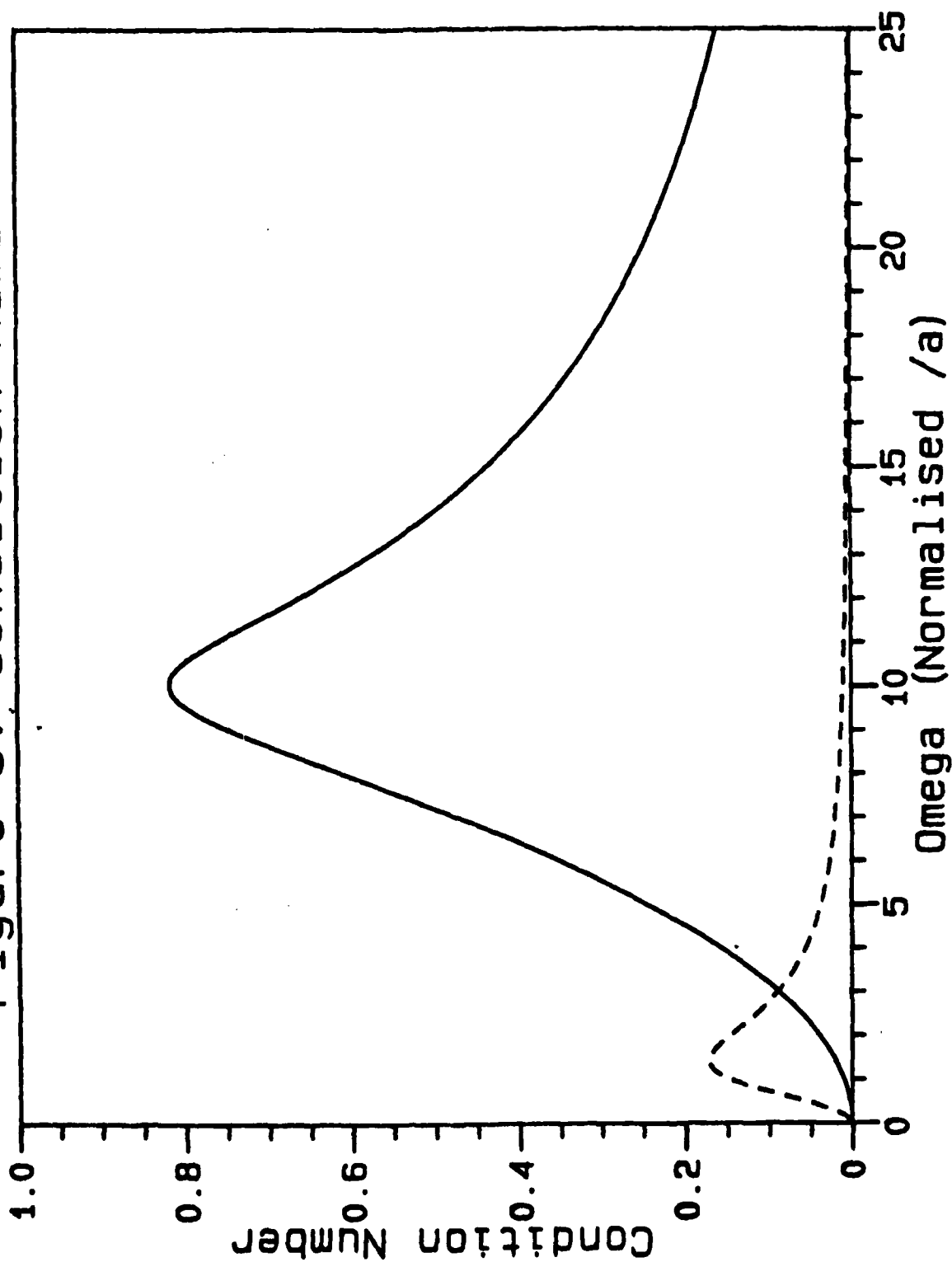
Figure 1. Determinant of Excitation Matrix

Figure 3. Condition Number

STABILITY THEORY FOR ADAPTIVE SYSTEMS:

METHOD OF AVERAGING AND PERSISTENCY

OF EXCITATION*

by

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ABSTRACT

A method of averaging is developed for the stability analysis of linear differential equations with small time-varying coefficients which do not necessarily possess an average value. The technique is then applied to determine the stability of a linear equation which arises in the study of adaptive systems where the adaptive parameters are slowly varying. The stability conditions are stated in the frequency-domain which shows the relation between persistent excitation and unmodeled dynamics.

1. INTRODUCTION

For a large class of adaptive feedback systems, as well as for some output error identification schemes, a stability analysis in the neighborhood of the desired behavior leads to investigating the stability of the following linear system of differential-operator equations (see e.g., [1]-[3], [20])

$$\dot{\theta} = \epsilon[f - \phi H(\phi'\theta)] \tag{1.1a}$$

where $\theta(0) = \theta_0 \in \mathbb{R}^p$, ϵ is a positive constant, $f(\cdot)$, $\phi(\cdot) : \mathbb{R}_+ \to \mathbb{R}^p$ are regulated and bounded, and H is a linear-time-invariant convolution operator with kernel $h(t)$ and transfer function $H(s)$, i.e.,

$$(Hu)(t) = \int_0^t h(t-\tau)u(\tau)d\tau \tag{1.1b}$$

We consider the case when $H(s)$ is strictly proper and exponentially stable, thus, $h(t)$ is bounded by a decaying exponential. The strictly proper assumption is not necessary for analysis, but it is more often the case when (1.1) arises from dynamical systems. The same can be said for considering the general convolution (1.1b) and not just the case of rational $H(s)$.

The specific problem we consider is slow adaptation (small $\epsilon > 0$), and to determine sufficient conditions for which the map $(f,\theta_0) \to \theta$ defined implicitly by (1.1), is exponentially stable, i.e., there are positive constants K, α such that

$$|\theta(t)| \le \int_0^t Ke^{-\alpha(t-\tau)}|f(\tau)|d\tau + Ke^{-\alpha t}|\theta_0| \tag{1.2}$$

When such a condition exists, it then follows that the adaptive systems from which (1.1) arose is locally stable.

Linearization and Local Stability

In [2], for example, system (1.1) is obtained as a result of linearization of the adaptive system in the neighborhood of a "tuned" system, i.e., a system where the adaptive parameters are set to a constant value $\theta_* \in \mathbb{R}^p$ and whose behavior is deemed acceptable. Hence, in (1.1), $\theta(t)$ is the vector of parameter errors between the

parameter estimate at time *t* and the tuned value θ∗, φ(*t*) is the regressor vector from the tuned system (e.g., filtered revisions of measured signals), and the scalar ε is the magnitude of the adaptation gain which essentially controls the rate of adaptation. The operator *H* depends on the actual system being controlled or identified and also on the tuned parameter setting θ∗.

It is shown in [2,3] that if system (1.1) is exponentially stable, then the adaptive system is locally stable, i.e., the adaptive system behavior will remain in a neighborhood of the desired behavior provided the initial parameter error θ(0) and the effect of external disturbances are sufficiently small. Although the results in [2,3] were arrived at using input-output properties [16], the local stability property also follows from the results on "total" stability [4], [20].

Unmodeled Dynamics and Slow Adaptation

In the ideal case there are a sufficient number of adaptive parameters (the number *p*) such that the tuned parameter setting results in $H(s)$ being strictly positive real (SPR), i.e., $\text{Re}H(j\omega) > 0$, ∀ ω ∈ $\mathbb{R}_+$. Under these conditions, we have the following results (see e.g., [5]-[8], [1]): (1) system (1.1) is stable, i.e., θ(*t*) is bounded but not necessarily constant; (2) if, in addition, φ(*t*) is persistently exciting, then system (1.1) is exponentially stable. The trouble starts when there are an insufficient number of parameters to obtain $H(s) \in$ SPR, as is the case in adaptive control when the plant has unmodeled dynamics (see e.g., [2,7], [12]).

In this paper we will examine the stability of (1.1) when ε is small, φ(*t*) is persistently exciting, and $H(s)$ is not necessarily SPR but only exponentially stable. Riedle and Kokotovic [9] refer to this case as "slow adaptation" and by using the method of averaging described by Hale [10], they show that the stability of (1.1) is critically dependent on the spectrum of the excitation in relation to the frequency response $H(j\omega)$. With the same assumptions, Astrom [11] uses averaging techniques to analyze the interaction between unmodeled dynamics and external inputs in the counter-example posed by Rohrs et al. [12]. Both these analyses require the assumption that φ(*t*) is almost periodic and that $H(s)$ is rational. In this case Riedle and Kokotovic [9] show system (1.1) is exponentially stable if

$$\lambda\left[\sum_{\omega \in \Omega} [\alpha(\omega)\alpha(\omega)^*]\text{Re}H(j\omega)\right] > 0 \qquad (1.3)$$

where Ω and $\{\alpha(\omega), \omega \in \Omega\}$ are, respectively, the Fourier exponents and coefficients of $\phi(t)$. Condition (1.2) can be considered as a *signal dependent positivity condition*, but unlike the SPR condition $\text{Re}H(j\omega)$ is not required to be positive at *all* frequencies.

The main contribution of this paper is to extend the theory of averaging to include the case when $\phi(t)$ does not have a (generalized) Fourier series representation, but is only known to be regulated and bounded. Thus, $\phi(t)$ need not be almost periodic nor even possess an average value. We also state stability conditions in the frequency-domain in a form similar to (1.2). Moreover, $H(s)$ need not be rational. Analogous results can be stated for discrete-time systems, see, e.g., [13].

Averaging: Uses and Limitations

The averaging theory developed here, as well as averaging theory in general, has its uses and limitations for adaptive system. In the first place, the theory requires slow adaptation which can be counter-productive because performance can be below par for the long period of time it takes for the parameters to adjust. Secondly, the averaging results developed in the sequel concern linear time-varying systems only, so that application of these results to the nonlinear adaptive system requires a linearization. In this sense we can obtain information, including frequency domain information, about the dynamical behavior of the adaptive system in the neighborhood of the tuned system. Both stability and instability conditions are discussed. The results arising from a combination of small gain theory and perturbation methods, e.g., [2, 3, 14, 15], are restricted to stability results, and are far less quantitative.

Organization of Paper

The paper is organized as follows: Section 2 develops a method of averaging for linear systems with *sample averages*. In Section 3 we apply the general results of Section 2 to (1.1) and obtain conditions for stability and instability. In Section 4 these are interpreted in terms of frequency domain stability conditions. In Section 5 we provide a general discussion.

Notation

The symbol $|\cdot|$ denotes both the vector norm as well as its induced matrix norm. Similarly, $\|\cdot\|_p$, $p \in [1,\infty]$, denotes the L_p-norm of a vector or matrix function, i.e., for

$$p \in [1,\infty), \quad \|F\|_p = (\int_0^\infty |F(t)|^p dt)^{1/p}, \quad \text{and} \quad \|F\|_\infty = \text{ess} \quad \sup \{|F(t)|: t \geq 0\}. \quad \lambda_i(A)$$

denotes the ith eigenvalue of matrix A and $\sigma_i(A)$ denotes the ith singular value of A, i.e., $\sigma_i(A) = [\lambda_i(A^*A)]^{1/2}$. An operator H is L_p-stable if $\exists$ constants k, b such that $\|Hu\|_p \leq k\|u\|_p + b$, $\forall$ $u \in L_p$. The smallest k is referred to as the L_p-gain, and is denoted by $\gamma_p(H)$.

2. METHOD OF AVERAGING FOR LINEAR HOMOGENEOUS SYSTEMS

In this section we will consider the homogeneous linear time-varying system

$$\dot{x} = \varepsilon A(t)x \tag{2.1}$$

Lemma 2.1:

Suppose in (2.1) that ε is a real constant and $A(\cdot): \mathbb{R}_+ \to \mathbb{R}^{n \times n}$ is regulated and bounded. Then $\forall$ $s, \tau \in \mathbb{R}_+$, the transition matrix $F(s+\tau, s)$ of (2.1) is given by

$$F(s+\tau, s) = \exp[\varepsilon \tau \bar{A}_\tau(s)] + R(s, \varepsilon \tau) \tag{2.2}$$

where

$$\bar{A}_\tau(s) = \frac{1}{\tau} \int_s^{s+\tau} A(t)dt \tag{2.3}$$

is referred to as the *sample average value* of $A(t)$ on the interval $s \leq t \leq s+\tau$, and

$$\|R(\cdot, \varepsilon\tau)\|_\infty \leq (\varepsilon\tau\|A\|_\infty)^2 \exp(\varepsilon\tau\|A\|_\infty) := r(\varepsilon\tau\|A\|_\infty) \tag{2.4}$$

Proof.

Using the Peano-Baker series representation for the transition matrix of (2.1) gives:

$$F(s+\tau, s) = I + \varepsilon \int_s^{s+\tau} A(t)dt + \sum_{k=2}^\infty \varepsilon^k \int_s^{s+\tau} A(t_1) \int_s^{t_1} A(t_2) \dots \int_s^{t_{k-1}} A(t_k)dt_1 \dots dt_k$$

Using definitions (2.2)-(2.3) for $R(s, \varepsilon\tau)$ and $\overline{A}_\tau(s)$, respectively, together with the series expansion for $\exp(\varepsilon\tau\overline{A}_\tau(s))$ results in,

$$R(s, \varepsilon\tau) = \sum_{k=2}^{\infty}[-(\varepsilon\tau\overline{A}_\tau(s))^k /k!$$

$$+ \varepsilon^k \int_s^{s+\tau} A(t_1) \int_s^{t_1} A(t_2) \dots \int_s^{t_{k-1}} A(t_k)dt_1 \dots dt_k]$$

$$\leq 2\sum_{k=2}^{\infty}(\varepsilon\tau\|A\|_\infty)^k/k! , \quad \forall \ s \in \mathbb{R}_+$$

$$= (\varepsilon\tau\|A\|_\infty)^2\exp(\varepsilon\tau\|A\|_\infty)$$

since $\|\overline{A}_\tau(\cdot)\|_\infty \leq \|A(\cdot)\|_\infty$. This proves (2.4).

$\square$

Remarks:

1. Assuming that $A(t)$ is regulated and bounded is sufficient for the existence and uniqueness of solutions [17].

2. Observe that Lemma 2.2 is valid $\forall \ s, \tau \in \mathbb{R}_+$ and $\forall \ \varepsilon \in \mathbb{R}$. In the sequel we use Lemma 2.2 only for the case when $\varepsilon > 0$ and $\varepsilon\tau$ is small.

The stability properties of (2.1) can be established by application of Lemma 2.2 as stated in Theorem 2.1 below. We first require:

Definition:

The function $\mu(\cdot): \mathbb{R}^{n \times n} \to \mathbb{R}$, defined by

$$\mu(M) = \lim_{\alpha \downarrow 0} (|I+\alpha M| - 1)/\alpha \tag{2.5}$$

is called the *measure of the matrix* M, where $|\cdot|$ is an induced matrix norm on $\mathbb{R}^{n \times n}$.

For any induced matrix norm and its corresponding measure, the following properties hold (see, e.g., [16]):

(P1) $\quad -|\,M\,| \le -\mu(-M) \le \text{Re}\lambda(M) \le \mu(M) \le |\,M\,|, \quad \forall \; M \in \mathbb{R}^{n \times n}$ (2.6a)

(P2) $\quad \mu(M_1 + M_2) \le \mu(M_1) + \mu(M_2), \quad \forall \; M_1, M_2 \in \mathbb{R}^{n \times n}$ (2.6b)

(P3) The transition matrix $F(t,\tau)$ of $\dot{x} = M(t)x$ satisfies,

$$\exp\left(-\int_\tau^t \mu[-M(s)]ds\right) \le |F(t,\tau)| \le \exp\left(\int_\tau^t \mu[M(s)]ds\right)$$ (2.6c)

(P4) If the vector norm on $\mathbb{R}^n$ is $|x| = (x'Px)^{1/2}$, $P = P' > 0$, then

$$|\,M\,| = \max_i \sigma_i(P^{1/2}MP^{-1/2})$$ (2.6d)

$$\mu(M) = \frac{1}{2} \max_i \lambda_i(P^{1/2}MP^{-1/2} + P^{-1/2}M'P^{1/2})$$ (2.6e)

These properties, together with Lemma 2.2, yield the following stability result for system (2.1).

Theorem 2.1:

Suppose $A(t)$ in (2.1) is regulated and bounded with the sequence of sample averages $\{\bar{A}_T(kT), \; \forall \; k \in \mathbb{Z}_+\}$. Then:

(i) If $\exists \; T > 0$ and $\alpha > 0$ such that

$$\mu[\bar{A}_T(kT)] \le -\alpha \;, \quad \forall \; k \in \mathbb{Z}_+$$ (2.7)

then $\exists \; \eta > 0$ such that $\forall \; \varepsilon T \in (0,\eta)$ the zero solution of (2.1) is u.a.s.

(ii) If $\exists \; T > 0$ and $\alpha > 0$ such that

$$\mu[-\bar{A}_T(kT)] \le -\alpha \;, \quad \forall \; k \in \mathbb{Z}_+$$ (2.8)

then $\exists \; \eta > 0$ such that $\forall \; \varepsilon T \in (0,\eta)$ the zero solution of (2.1) is completely unstable.

Proof.

Combining (2.6c) with (2.2) gives,

$$|\exp[-\varepsilon\tau\mu(-\bar{A}_\tau(s))]| \le |F(s+\tau, s) - R(s,\varepsilon\tau)| = |\exp[\varepsilon\tau\bar{A}_\tau(s)]|$$

$$\le \exp[\varepsilon\tau\mu(\bar{A}_\tau(s))] \;, \quad \forall \; s,\tau \in \mathbb{R}_+$$

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which implies,

$$|F(s+\tau, s)| \leq \exp[\varepsilon\tau\mu(\overline{A}_\tau(s))] + r(\varepsilon\tau m) \tag{2.9}$$

$$|F(s+\tau, s)| \geq \exp[-\varepsilon\tau\mu(\overline{A}_\tau(s))] - r(\varepsilon\tau m) \tag{2.10}$$

where we have used (2.4) with $\|A\|_\infty = m$.

We first prove part (i) by using condition (2.7) and inequality (2.9) with $\tau = T$ and $s = kT$. This gives,

$$|F((k+1)T, kT)| \leq \exp(-\varepsilon T\alpha) + r(\varepsilon T m) \quad , \quad \forall \ k \in \mathbf{Z}_+$$

We now establish that for all small $\varepsilon T > 0$, $|F((k+1)T, kT)| < 1$, i.e., the map $\theta(kT) \to \theta((k+1)T)$ is a contraction. From the definition of $r(\cdot)$ in (2.4), it follows that for any $\alpha > 0$ there is a $\eta > 0$ such that

$$\exp(-\eta\alpha) + r(\eta m) = 1 \tag{2.11}$$

Hence, for all $\varepsilon T \in (0,\eta)$, there is a $\beta > 0$ such that

$$\exp(-\varepsilon T\alpha) + r(\varepsilon T m) = \exp(-\varepsilon T\beta) < 1 \tag{2.12}$$

which shows the contraction property.

Now, for any $t,s \in \mathbb{R}_+$ with $t \geq s$, there exists an integer $k \geq 0$ such that $s+kT \leq t \leq s+(k+1)T$. Thus,

$$\begin{aligned}
|F(t,s)| &= |F(t,s+kT)F(s+kT, s+(k-1)T) \dots F(s+T, s)| \\
&\leq |F(t,s+kT)|\exp(-\varepsilon kT\beta) \quad , \\
&\leq |F(t,s+kT)|\exp(-\varepsilon(t-s-T)\beta) \quad , \quad \text{by } kT \geq t-s-T \\
&\leq \exp(\varepsilon T(m+\beta))\exp(-\varepsilon(t-s)\beta)
\end{aligned}$$

The last line follows from Property (2.6c), i.e.,

$$\begin{aligned}
|F(t,s+kT)| &\leq \exp\left(\int_{s+kT}^{t} \mu[\varepsilon A(\tau)]d\tau \right) \\
&\leq \exp(\varepsilon m(t-s-kT)) \quad , \quad \text{by } \mu[A(\tau)] \leq |A(\tau)| \leq m \\
&\leq \exp(\varepsilon mT) \quad , \quad \text{by } t-s-kT \in (0,T)
\end{aligned}$$

This proves part (i) of Theorem 2.1. The proof of part (ii) follows from the above analysis, but starting with inequality (2.10).

□

Using the same technique, but allowing $A(\cdot)$ (equivalently $\overline{A}_T(\cdot)$) to possess a uniform average, we obtain the following sharper result.

Theorem 2.2:

Suppose $A(t)$ in (2.1) is regulated, bounded, and has a *uniform average* $\overline{A} \in \mathbb{R}^{n \times n}$, i.e.,

$$\lim_{T \to \infty} \overline{A}_T(s) = \overline{A} \qquad (2.13)$$

uniformly $\forall\ s \in \mathbb{R}$. Under these conditions:

(i) If $\exists\ \alpha > 0$ such that

$$\text{Re}\lambda(\overline{A}) \leq -\alpha \qquad (2.14)$$

then $\exists\ \varepsilon_0 > 0$ such that $\forall\ \varepsilon \in (0, \varepsilon_0)$ the zero solution of (2.1) is u.a.s.

(ii) If $\exists\ \alpha > 0$ such that $\text{Re}\lambda(\overline{A}) \neq 0$ and

$$\max \text{Re}\lambda(\overline{A}) \geq \alpha \qquad (2.15)$$

then $\exists\ \varepsilon_0 > 0$ such that $\forall\ \varepsilon \in (0, \varepsilon_0)$, the zero solution of (2.1) is unstable.

Proof.

We first prove part (i). Assumption (2.13) means that $\forall\ \delta > 0$, $\exists\ T(\delta) > 0$ such that

$$|\overline{A}_T(s) - \overline{A}| \leq \delta \ , \qquad \forall\ s \in \mathbb{R}_+ \qquad (2.16)$$

From (2.9), with $\|A(\cdot)\|_\infty = m$, we have

$$|F(s + T, s)| \leq \exp[\varepsilon T \mu(\overline{A} + \overline{A}_T(s) - \overline{A}] + (\varepsilon T m)^2 \exp(\varepsilon T m)$$

$$\leq \exp[\varepsilon T(\mu(\overline{A}) + \delta)] + (\varepsilon T m)^2 \exp(\varepsilon T m)$$

Since $\text{Re}\lambda(\overline{A}) < 0$, there is a constant matrix $P = P' > 0$ which satisfies the

Lyapunov equation,

$$\bar{A}'P + P\bar{A} + 2I = 0 \tag{2.18}$$

Now, choose as a norm on $\mathbb{R}^n$,

$$|x| = (x'Px)^{1/2} \tag{2.19}$$

From (2.6d) and using (2.18) we then have,

$$\mu(\bar{A}) = -\frac{1}{2} \max_i \lambda_i\{P^{-1/2}(\bar{A}'P + P\bar{A})P^{-1/2}\}$$

$$= -\min_i \lambda_i\{P^{-1}\} = -\alpha \tag{2.20}$$

Hence, (2.17) becomes,

$$|F(s + \tau, s)| \leq \exp[-\varepsilon T(\alpha - \delta)] + (\varepsilon TM)^2 \exp(\varepsilon TM) \tag{2.21}$$

By assumption (2.13) it is always possible to select $T(\delta)$ in (2.16) such that $\delta < \alpha$. By inspection of (2.21), there then exists $\varepsilon_0 > 0$ such that $\forall \, \varepsilon \in (0, \varepsilon_0)$, $|F(s + T, s)| < 1$, $\forall \, s \in \mathbb{R}_+$, which completes the proof of part (i). Part (ii) can be proven in an analogous manner starting with (2.10) and using (2.18) with $\bar{A}$ replaced by $-\bar{A}$. Note that here Re $\lambda(\bar{A}) \neq 0$ is required explicitly.

$$\square$$

Discussion

The results in Theorem 2.9 and Theorem 2.2 generalize some results obtained by averaging methods such as those described by Hale [10], or as obtained by Coppel [18] using the notion of integral smallness. Theorem 2.2 is a classical result of averaging theory, except that as stated it allows for functions which are not necessarily almost periodic. The class of functions allowed in Theorem 2.2 -- regulated, bounded, with a uniform average -- is not precisely characterized. Obviously it includes the class of asymptotically almost periodic functions of the form

$$A(t) = A_0(t) + A_1(t) \tag{2.22}$$

where $A_0(t)$ is almost periodic and $A_1(\cdot) \in L_1^{n \times n}$, $p \in [1, \infty]$.

Theorem 2.1 considers a larger class of functions -- those without an average -- at the expense of a weaker result: the stability-instability boundary is not as sharp as in

Theorem 2.2.

An example of a function which satisfies the conditions of Theorem 2.1, but not of Theorem 2.2 is:

$$A(t) = A_0 + (1/\sqrt{2})A_1(\sin \log t + \cos \log t) \tag{2.23}$$

where $A_i = A_i' > 0$, $i = 0, 1$ such that $A_0 - A_1 > 0$. This function does not have a uniform average, as can be seen from

$$\frac{1}{T}\int_s^{s+T} A(t)dt = A_0 + \frac{A_1}{\sqrt{2}}\left[\frac{s+T}{T}\sin \log(s+T) - \frac{s}{T}\sin \log s\right] \tag{2.24}$$

However, it satisfies the conditions of Theorem 2.1 because from (2.23)

$$\frac{1}{T}\int_s^{s+T} A(t)dt \geq A_0 - A_1 > 0 \ , \quad \forall \ s \in \mathbb{R}_+, \ \forall \ T > 0 . \tag{2.25}$$

Condition (2.7), which is the basis for the u.a.s. property, has some interesting interpretations. In the first place, since $\alpha > 0$ is a constant, conditions (2.10) provides a uniform bound on the sequence of sample-average measures $\{\mu[\bar{A}_T(kT)], k \in \mathbb{Z}_+\}$. From the definition (2.5), the measure is dependent on the underlying vector norm. Suppose we choose as the vector norm $|x| = (x'Px)^{1/2}$ with $P = P' > 0$ a constant matrix. This was done in the proof of Theorem 2.2 where P was given as the solution to (2.18). In general, however, we have from (2.6e) that

$$\mu[\bar{A}_T(kT)] = \frac{1}{2}\max_i \lambda_i\{P^{-1/2}[\bar{A}_T'(kT)P + P\bar{A}_T(kT)]P^{-1/2}\} \tag{2.26}$$

If there is a constant matrix $P = P' > 0$ such that

$$\frac{1}{2}\max_i \lambda_i\{\bar{A}_T'(kT)P + P\bar{A}_T(kT)\} \leq -1 \ , \quad \forall \ k \in \mathbb{Z}_+ \tag{2.27}$$

then (2.7) holds with the choice

$$\alpha = \min_i \lambda_i(P^{-1}) \tag{2.28}$$

Observe that (2.27) is *not* equivalent to

$$\text{Re } \lambda\{\bar{A}_T(kT)\} < 0 \ , \quad \forall \ k \in \mathbb{Z}_+ \tag{2.29}$$

This latter condition means there is a *sequence* of matrices $\{\bar{P}(k) = \bar{P}(k)' > 0, k \in \mathbb{Z}_+\}$ which satisfy

$$\max_i \lambda_i\{\bar{A}_T{}'(k)\bar{P}(k) + \bar{P}(k)\bar{A}_T(kT)\} = -2 \quad , \quad \forall \ k \in Z_+ \tag{2.30}$$

Unfortunately, it makes no sense to choose a time-varying norm, e.g., $|x| = (x'\bar{P}(k)x)^{1/2}$. Hence, condition (2.27) provides a means to satisfy (2.7), provided that *constant* P can be found.

A simple sufficient condition for (2.20) is that

$$\max_i \lambda_i\{\bar{A}(k) + \bar{A}(k)'\} \leq -2\alpha_0 \quad , \quad \forall \ k \in Z_+ \tag{2.31}$$

where α_0 is a positive constant. Hence, we can take $P = (1/\alpha_0)I$ in (2.27) and thus (2.10) holds with $\alpha = \alpha_0$. We will discuss condition (2.31) further when we specialize Theorem 2.1 for adaptive systems in Section 4.

Theorem 2.1 also requires that $\varepsilon T > 0$ be sufficiently small, i.e., that $\varepsilon T \in (0,\eta)$. From the proof of Theorem 2.1 we can extract a value for η and also state bounds on the exponential rates of growth or decay of the transition matrix $F(t,\tau)$ for all $t \geq \tau$. Specifically, we have:

Corollary 2.1:

If $A(t)$ is regulated and bounded with $\|A(\cdot)\|_\infty \leq m$, then:

(i) Whenever (2.10) holds for some $T > 0$, the zero solution of (2.1) is u.a.s. $\forall \ \varepsilon T \in (0,\eta)$, i.e.,

$$|F(t,\tau)| \leq M \exp(-\varepsilon(t-\tau)\beta) \tag{2.32}$$

where $\eta, M,$ and β satisfy:

$$\exp(-\eta\alpha) + r(\eta m) = 1$$

$$M = \exp(\varepsilon T(m + \beta)) > 1$$

$$\exp(-\varepsilon T\beta) = \exp(-\varepsilon T\alpha) + r(\varepsilon Tm) < 1 \tag{2.33}$$

(ii) Whenever (2.11) holds for some $T > 0$, the zero solution of (2.1) is unstable $\forall \ \varepsilon T \in (0,\eta)$, i.e.,

$$|F(t,\tau)| \geq M \exp(\varepsilon(t-\tau)\beta) \tag{2.34}$$

where $\eta, M,$ and β satisfy

$$\exp(\eta\alpha) - r(nm) = 1$$

$$M = \exp(-\varepsilon T(m + \beta)) < 1 \qquad (2.35)$$

$$\exp(\varepsilon T\beta) = \exp(\varepsilon T\alpha) - r(\varepsilon Tm) < 1$$

☐

3. STABILITY OF LINEARIZED ADAPTIVE SYSTEM

In this section we apply the results of Section 2 to the linearized adaptive system (1.1) under slow adaptation, i.e., small $\varepsilon > 0$. The first step is to transform (1.1) into a form suitable for application of Theorem 2.1. This is accomplished by a *time-scale decomposition*. That is, under slow adaptation the parameters $\theta(t)$ change much more slowly than the internal states of the dynamical system H. This suggests approximating (1.1) by the system

$$\dot{\theta} = \varepsilon[f - (\phi H\phi')\theta] \qquad (3.1)$$

for which Theorem 2.1 would apply, i.e., replace $A(t)$ in (2.1) with $-\varepsilon(\phi H\phi')(t)$. We start with the following intermediate result, developed in [20] and based on the discrete-time formulation in [21].

Lemma 3.1:

System (1.1) is equivalent to

$$\dot{\theta} = \varepsilon[f - R\theta + \varepsilon W(f,\theta)] \qquad (3.2)$$

where R is the time varying matrix

$$R(t) = (\phi H\phi')(t) \qquad (3.3)$$

and $W(f,\theta)$ is the linear integral operator

$$W(f,\theta) = \phi G_\phi[f - \phi H(\phi'\theta)] \qquad (3.3)$$

with G_ϕ the linear integral operator whose kernel is,

$$g_\phi(t,\tau) = \int_0^\tau h(t-s)\phi'(s)ds \quad , \quad 0 \le \tau \le t \qquad (3.5)$$

Proof.

Integrating by parts gives

$$H(\phi'\theta) = (H\phi')\theta - G_\phi\dot{\theta}$$

Thus,

$$W(f,\theta) = \frac{1}{\varepsilon}\,\phi[(H\phi')\theta - H(\phi'\theta)]$$

$$= \frac{1}{\varepsilon}\,\phi G_\phi\dot{\theta} \qquad \text{by (3.5)}$$

$$= \phi G_\phi[f - \phi H(\phi'\theta)]$$

by (1.1).

$$\square$$

Discussion

The decomposition of (1.1) into (3.2) is illustrated by the feedback system:

Figure 3.1

Hence, for small $\varepsilon > 0$, the operator $\varepsilon W(f,\theta)$ has little effect on system stability, and $\dot{\theta} = \varepsilon(f-R\theta)$ provides the dominating stabilizing force. We will prove this assertion in Theorem 3.1 below.

If $H(s)$ is rational, i.e., $\exists\, A \in \mathbb{R}^{n \times n}$ and $b, c \in \mathbb{R}^n$ such that

$$H(s) = c'(sI - A)^{-1}b \qquad\qquad$$

then the decomposition (3.2) is essentially equivalent to the L–transformation in [9a,b],

which is also a Lyapunov transformation, i.e., both original and transformed systems have identical (Lyapunov) stability properties, see [24, p. 117]. In this case system (3.1) is equivalently represented in state form as

$$\dot{\theta} = \varepsilon(f(t) - \phi(t)c'z) \quad , \quad \theta(0) = \theta_0 \tag{3.7a}$$

$$\dot{z} = Az + b\phi'(t)\theta \quad , \quad z(0) = 0 \tag{3.7b}$$

Using the "L-transformation"

$$\xi = z - L(t)\theta \tag{3.8a}$$

where $L(t)$ satisfies

$$\dot{L} = AL + b\phi'(t) \tag{3.8b}$$

gives

$$\dot{\theta} = \varepsilon[f(t) - \phi(t)c'(L(t)\theta + \xi)] \tag{3.9a}$$

$$\dot{\xi} = A\xi - \varepsilon L(t)[f(t) - \phi(t)c'(L(t)\theta + \xi)] \tag{3.9b}$$

Since $\theta(0) = \theta_0$ and $z(0) = 0$ by definition (1.1), it follows that by assigning $L(0) = 0$ we have[*]

$$R(t) = \phi(t)c'L(t) \tag{3.10}$$

and hence, (3.9a) becomes

$$\dot{\theta} = \varepsilon[f(t) - R(t)\theta - \phi(t)c'\xi] \tag{3.11}$$

Since $\xi(0) = 0$ from (3.8) and Re $\lambda(A) < 0$ because $H(s)$ is stable, it follows that $\xi(t) = O(\varepsilon)$. Thus, (3.11) is dominated for small ε by $\dot{\theta} = \varepsilon(f - R\theta)$. Consequently, both the "L-transformation" and the operator decomposition (3.2) are qualitatively equivalent for small ε.

Using Lemma 3.1, we now state conditions for exponential stability of system (1.1), i.e., the map $(\theta_0, f) \rightarrow \theta$.

---

[*] Note that the choice of initial condition for $L(0)$ is immaterial when discussing asymptotic stability properties, i.e., since A is stable, different initial conditions give rise to different exponentially fast decaying transients.

Theorem 3.1:

Assume that:

(A 1) $\dot{\theta} = -\varepsilon R(t)\theta$, $R(t) = (\phi H\phi')(t)$, is u.a.s. with transition matrix $F(t,\tau)$ overbounded by

$$|F(t,\tau)| \leq M\, e^{-\varepsilon\beta(t-\tau)} \quad , \quad \forall\ t \geq \tau \geq 0 \tag{3.12}$$

(A 2) The impulse response $h(t)$ of H satisfies

$$|h(t)| \leq K\, e^{-\alpha t} \quad , \quad \forall\ t \geq 0 . \tag{3.13}$$

Under these conditions, $\exists\ \varepsilon_0 > 0$ such that $\forall\ \varepsilon \in (0,\varepsilon_0)$, system (1.1) is exponentially stable. Specifically, if

$$\varepsilon_0 = \min\{\alpha/\beta, \varepsilon_1\} \quad , \quad \rho(\varepsilon_1) = 0 \tag{3.14}$$

then

$$|\theta(t)| \leq M|\,\theta_0\,|e^{-\varepsilon\rho(\varepsilon)t} + \int_0^t \varepsilon m(\varepsilon)e^{-\varepsilon\rho(\varepsilon)(t-\tau)}|f(\tau)|d\tau \tag{3.15}$$

where

$$m(\varepsilon) = M[1 + \varepsilon\|\phi\|_\infty^2 K/(\alpha - \varepsilon\beta)^2] \tag{3.16}$$

$$\rho(\varepsilon) = \beta - \varepsilon M\|\phi\|_\infty^4 K^2/(\alpha - \varepsilon\beta)^2 \tag{3.17}$$

Proof.

Using the decomposition from Lemma 3.1 gives the following expression for (1.1):

$$\theta(t) = F(t,0)\theta_0 + \varepsilon(W_1 f)(t) - \varepsilon^2(W_2\theta)(t) \tag{3.18}$$

where W_1, W_2 are linear integral operators given by

$$W_1 = F(I + \varepsilon\phi G_\phi) \tag{3.19a}$$

$$W_2 = F\phi G_\phi\phi H\phi' \tag{3.19b}$$

and where F has kernel $F(t,\tau)$. We first show that W_1 and W_2 are exp. stable integral operators.

For *any* integral operator W with kernel $w(t,\tau)$ we have

$$(Wu)(t) = \int_0^t w(t,\tau)u(\tau)d\tau$$

$$= \int_0^t e^{-\sigma(t-\tau)}[e^{\sigma(t-\tau)}w(t,\tau)]u(\tau)d\tau$$

Therefore, W is exp. stable iff $\exists \ \sigma > 0$ such that

$$\sup_{t \geq \tau} |e^{\sigma(t-\tau)}w(t,\tau)| < \infty$$

Let the superscript notation $(\cdot)^\sigma$ denote *exponential weighting*, i.e., $(x^\sigma)(t) = e^{\sigma t}x(t)$. Hence, $(Wu)^\sigma = W^\sigma u^\sigma$ where W^σ is the linear integral operator with kernel $e^{\sigma(t-\tau)}w(t,\tau)$. Now, following pg. 119 of [16], let $\|W\|_b$ be defined by

$$\|W\|_b = \sup_{t \geq \tau} |w(t,\tau)| \tag{3.20}$$

Hence, W is exp. stable iff $\exists \ \sigma > 0$ such that

$$\|W^\sigma\|_b < \infty$$

Observe also that if G_1 and G_2 are linear integral operators then

$$\|G_1 G_2\|_b \leq \|G_1\|_b \gamma_1(G_2) \tag{3.21}$$

Applying these relations to (3.19) for some $\sigma > 0$ gives,

$$\|W_1^\sigma\|_b \leq \|F^\sigma\|_b[1 + \epsilon\|\phi\|_\infty \gamma_1(G_\phi^\sigma)]$$

$$\|W_2^\sigma\|_b \leq \|F^\sigma\|_b\|\phi\|_\infty^3 \gamma_1(G_\phi^\sigma)\gamma_1(H^\sigma)$$

Choose $\sigma = \epsilon\beta$ with $\epsilon \in (0,\epsilon_0)$, ϵ_0 given by (3.14). Using (3.19)-(3.21) gives

$$\|F^\sigma\|_b \leq M$$

$$\gamma_1(H^\sigma) \leq K/(\alpha - \epsilon\beta)$$

$$\gamma_1(G_\phi^\sigma) \leq \|\phi\|_\infty K/(\alpha - \epsilon\beta)^2$$

Hence,

$$\|W_1^\sigma\|_b \leq M[1+\epsilon\|\phi\|_\infty^2 K/(\alpha - \epsilon\beta)^2] = m(\epsilon)$$

$$\|W_2^\sigma\|_b \leq M\|\phi\|_\infty^4 K^2/(\alpha - \epsilon\beta)^3 = [\beta - \rho(\epsilon)]/\epsilon$$

where $m(\varepsilon)$ and $\rho(\varepsilon)$ are defined in (3.16), (3.17). Going back to (3.18), we now have,

$$|\theta(t)| \leq Me^{-\varepsilon\beta t}\,|\,\theta_0\,| + \int_0^t e^{-\varepsilon\beta(t-\tau)}\{\varepsilon m(\varepsilon)|f(\tau)|$$

$$+ \ \varepsilon[\beta-\rho(\varepsilon)]|\theta(\tau)|\}d\tau \tag{3.22}$$

The result (3.15)-(3.17) follows by directly applying the Bellman-Gronwall Lemma to (3.22).

$$\square$$

Discussion

Under slow adaptation, Theorem 3.1 shows that (1.1) is exponentially stable if $\dot{\theta} = -\varepsilon R(t)\theta$ is u.a.s. Hence, we can apply Theorem 2.1, with $A(t)$ replaced by $-\varepsilon R(t)$, and arrive at stability condition (2.7), that is:

System (1.1) is exponentially stable for all small $\varepsilon > 0$ *if* $\exists\ T > 0$ *such that*

$$\mu[-\bar{R}(k)] < 0 \quad , \quad \forall\ k \in \mathbf{Z}_+ \tag{3.23}$$

where $\bar{R}(k)$ *is the* k^{th} *sample average*

$$\bar{R}(k) = \frac{1}{T}\int_{kT}^{(k+1)T} R(t)dt \tag{3.24}$$

Using (2.27), condition (3.23) holds if there is a constant matrix $P = P' > 0$ such that

$$\frac{1}{2}\min_i \lambda_i\{\bar{R}(k)P + P\bar{R}(k)'\} \geq 1 \quad , \quad \forall\ k \in \mathbf{Z}_+ \tag{3.25}$$

Moreover, a sufficient condition for (3.23) is that

$$\min_i \lambda_i\{\bar{R}(k) + \bar{R}(k)'\} > \alpha_0 \quad , \quad \forall\ k \in \mathbf{Z}_+ \tag{3.26}$$

where α_0 is a positive constant. Comparing (3.26) to (3.25) reveals that $P = (1/\alpha_0)I$, which means the interval contraction of $\theta(kT) \to \theta((k+1)T)$ is scaled uniformly, i.e., $\theta'(kT+T)\theta(kT+T) < \theta'(kT)\theta(kT)$. The scaling implications are discussed further in Section 4 to follow.

4. FREQUENCY-DOMAIN STABILITY CONDITIONS

In this section we reformulate condition (3.12) in the frequency domain. This involves the Fourier transform $H(j\omega)$ and an appropriately defined expression for the spectrum of $\phi(t)$. We show that (3.12) requires that $\phi(t)$ have a persistent excitation property, and that the dominant excitation be at those frequencies for which $\text{Re } H(j\omega) > 0$.

The first requirement is that $\phi(t)$ be restricted to those functions which have a Fourier series representation on any finite interval. A known class of such functions is defined as follows (see, e.g., [19]).

Definition:

A function $f(\cdot)$: $\mathbb{R}_+ \to \mathbb{R}^n$ is a C_δ^r function if it is regulated, bounded and $\exists$ a constant $\delta > 0$ such that any two points $t_1, t_2 \in \mathbb{R}_+$ where $f(\cdot)$ is discontinuous are separated by at least an interval δ, i.e., $|t_1 - t_2| \geq \delta$.

Frequency-domain stability conditions for the stability of (3.1) can now be stated.

Theorem 4.1:

Assume in (1.1) that:

(A 1) $|h(t)| \leq Ke^{-\alpha t}$, $\forall$

$$t \geq 0. \tag{4.1}$$

(A 2) $\phi \in C_\delta^r$ with piece-wise Fourier series representation $\forall$ $k \in \mathbb{Z}_+$:

$$\phi(t) \sim \sum_{\omega \in \Omega_k} \alpha_k(\omega)e^{j\omega t} \quad , \quad \forall \; t \in [kT,(k+1)T) \quad , \quad T \geq \delta \tag{4.2}$$

where Ω_k is the set of distinct Fourier exponents and $\alpha_k(\cdot)$ the corresponding Fourier coefficients. Let $B(k) \in \mathbb{R}^{p \times p}$ be defined by[t]

$$B(k) = \sum_{\omega \in \Omega_k} \alpha_k(\omega)\overline{\alpha}_k(\omega)'H(-j\omega) \quad , \quad \forall \; k \in \mathbb{Z}_+ \tag{4.3}$$

---

[t] Overbar $(\bar{\cdot})$ denotes complex conjugation.

Under these conditions:

(i)   If $\exists\ T \geq \delta$ such that

$$\mu[-B(k)] \;<\; -2\|\phi\|_\infty^2\, (K/\alpha^2)/T \quad,\qquad \forall\ k \in \mathbb{Z}_+ \tag{4.4}$$

then $\exists\ \varepsilon_0 > 0$ such that $\forall\ \varepsilon \in (0,\varepsilon_0)$, system (1.1) is exponentially stable.

(ii)   If $\exists\ T \geq \delta$ such that

$$\mu[B(k)] \;<\; -2\|\phi\|_\infty^2\, (K/\alpha^2)/T \quad,\qquad \forall\ k \in \mathbb{Z}_+ \tag{4.5}$$

then $\exists\ \varepsilon_0 > 0$ such that $\forall\ \varepsilon \in (0,\varepsilon_0)$, system (1.1) is unstable.

Remarks:

1. The representation (4.2) for $\phi(t)$ specifies the local frequency content over $t \in [kT, (k+1)T]$. Such a representation -- if not given -- can always be found if $\phi(t) \in C_0^\infty$ [17]; then (4.2) can be obtained via the Fourier series of the T-periodic function:

$$\phi_k(t) \;=\; \phi(t + mT) \quad t \in [(k-m)T, (k-m+1)T]$$

$$\forall\ k \in \mathbb{N}\ ,\quad \forall\ m \in \mathbb{Z}$$

Notice $\phi_k(t)$ is well-defined on $\mathbb{R}$ and has a Fourier series representation:

$$\phi_k(t) \;\sim\; \sum_{m \in \mathbb{Z}} \alpha_k(j\omega_m)e^{j\omega_m t} \quad;\qquad \forall\ t \in \mathbb{R}$$

where

$$\alpha_k = \alpha_{-k}^* \quad,\qquad \omega_m = 2\pi m/T$$

and hence

$$\phi_k(t) \;\sim\; \sum_{m \in \mathbb{Z}} \alpha_k(j\omega_m)e^{j\omega_m t} \quad;\qquad \forall\ t \in [kT,(k+1)T] \tag{4.6}$$

which is of the form (4.2).

2. The matrix $B(k)$ can be equivalently expressed as the sample average value of the T-periodic part of $(\phi_k H \phi_k')(t)$, i.e.,

$$B(k) \;=\; \frac{1}{T} \int_{kT}^{(k+1)T} \phi_k(t)\psi_k(t)'\,dt \tag{4.7}$$

where $\psi_k(t)$ is the T-periodic part of $H(\phi_k)(t)$, i.e.,

$$\psi_k(t) = \int\limits_{-\infty}^{t} h(t-\tau)\phi_k(\tau)d\tau$$

$$= \sum_{\omega \in \Omega_k} H(j\omega)\alpha_k(\omega)e^{j\omega t} \quad , \quad \forall\ t \in \mathbb{R} \tag{4.8}$$

Proof.

To prove part (i), it follows from Lemma 3.1 and Theorem 3.1 that it is only necessary to show that (4.4) implies

$$\mu[-\bar{R}(k)] < 0 \quad , \quad \forall\ k \in \mathbb{Z}_+$$

where $\bar{R}(k)$ is given by (3.24). We start by defining

$$\tilde{B}(k) = \bar{R}(k) - B(k)$$

with $B(k)$ from (4.3). Using (4.8) gives

$$\tilde{B}(k) = \frac{1}{T} \int\limits_{kT}^{(k+1)T} \phi_k(t)\tilde{\psi}_k'(t)dt$$

where

$$\tilde{\psi}_k(t) = (H\phi)(t) - \psi_k(t)$$

$$= \int\limits_{0}^{t} h(t-\tau)\phi(\tau)d\tau - \int\limits_{-\infty}^{t} h(t-\tau)\phi_k(\tau)d\tau$$

$$= \int\limits_{0}^{kT} h(t-\tau)\phi(\tau)d\tau - \int\limits_{-\infty}^{kT} h(t-\tau)\phi_k(\tau)d\tau$$

The last line follows from (4.6), i.e., $\phi_k(t) = \phi(t)$ for $t \in [kT,(k+1)T]$. Using (4.1) gives,

$$|\tilde{\psi}_k(t)| \leq (2\|\phi\|_\infty K/\alpha)e^{-\alpha t}$$

from which it follows that

$$|\tilde{B}(k)| \leq 2\|\phi\|_\infty^2 (K/\alpha^2)/T \quad , \quad \forall\ k \in \mathbb{Z}_+$$

This together with inequality (2.6a) proves part (i). Part (ii) follows analogously by

replacing $\bar{R}(k)$ with $-\bar{R}(k)$.

□

If $\phi(t)$ is further restricted so that it has a uniform average, then we can sharpen the stability-instability boundary. For example, if $\phi(t)$ is almost periodic then a Fourier series representation exists $\forall\ t \in \mathbb{R}_+$, and thus, it has an average [10]. The stability conditions for this case are stated as follows.

Theorem 4.2:

Suppose in (1.1) that $\phi(t)$ is almost periodic with generalized Fourier series

$$\phi(t) \sim \sum_{\omega \in \Omega} \alpha(\omega)e^{j\omega t}\ , \qquad \forall\ t \in \mathbb{R}_+ \tag{4.9}$$

where $\Omega \in \mathbb{R}$ are the distinct Fourier exponents and $\{\alpha(\omega), \omega \in \Omega\}$ are the Fourier coefficients. Define the matrix B by

$$B = \sum_{\omega \in \Omega} \alpha(\omega)\bar{\alpha}(\omega)'H(-j\omega) \tag{4.10}$$

If $\text{Re}\ \lambda(B) \neq 0$ then $\exists\ \varepsilon_0 > 0$ such that $\forall\ \varepsilon \in (0,\varepsilon_0)$, system (1.1) is:

(i) exponentially stable if $\text{Re}\ \lambda(B) < 0$ \hfill (4.11)

(ii) unstable if $\max_{i} \text{Re}\ \lambda_i(B) > 0$ \hfill (4.12)

Remark: The proof of Theorem 4.2 is entirely analogous to that of Theorem 4.1. Theorem 4.2 is the result obtained in [9] when $\phi(t)$ is almost periodic. Theorem 4.1 is a generalization to $\phi(\cdot) \in C_b^\varepsilon$.

5. DISCUSSION OF RESULTS

(A) Effect of Transients on Sample Average

An informative interpretation of stability condition (4.4) is that the average energy in the T-periodic part of $(\phi_k H\phi_k')(t)$ must dominate (or overcome) the possibly negative efforts of the transient terms. In other words, the period T must be sufficiently larger than the dominant time constant of H, i.e., $T \gg 1/\alpha$. Note that the term $2\|\phi\|_\infty^2(K/\alpha^2)$ essentially arises from initial conditions or stored-energy in H at

$t = kT$. Obviously when $\phi(t)$ has a uniform average it is always possible to select T to be sufficiently large, e.g., as shown in the proof of Theorem 2.2.

Using (2.27) condition (4.4) holds if there is a constant matrix $P = P > 0$ such that $\forall\ k \in \mathbb{Z}_+$,

$$\min_i \lambda_i[Q(k)] \geq 1 \quad , \quad \forall\ k \in \mathbb{Z}_+ \tag{5.1a}$$

where

$$Q(k) = \frac{1}{2} \sum_{\omega \in \Omega_+} H(-j\omega)[PX_k(\omega) + \bar{X}_k(\omega)P] \tag{5.1b}$$

$$X_k(\omega) = \alpha_k(\omega)\bar{\alpha}_k(\omega)' = [\bar{X}_k(\omega)]' \tag{5.1c}$$

(B) Relation to Persistent Excitation

A necessary condition for the existence of P which satisfies (5.1) is that for some finite integer $q \geq (p-1)/2$ and $\forall\ k \in \mathbb{Z}_+$,

$$\text{rank}[\alpha_k(0), \alpha_k(\omega_1),..., \alpha_k(\omega_q), \bar{\alpha}_k(\omega_1),..., \bar{\alpha}_k(\omega_q)] = p \tag{5.2}$$

If this were not the case then $\min_i \lambda_i[Q_k(P)] = 0$, $\forall\ k \in \mathbb{Z}_+$ and $\forall\ P = P' > 0$. Hence, Theorem 4.1 implicitly restricts $\phi(\cdot) \in C_\delta^\tau$ to those functions whose (time-varying) Fourier coefficients satisfy the rank condition above. This class of functions, however, are precisely those which can be categorized as persistently exciting [1]:

Definition:

A function $f(\cdot): \mathbb{R}_+ \to \mathbb{R}^n$ is *persistently exciting* (PE) over an interval h if it is regulated, bounded, and $\exists$ constants $h > 0$ and $\beta > 0$ such that

$$\min_i \lambda_i \left[\frac{1}{h} \int_s^{s+h} f(t)f(t)'dt \right] \geq \beta \quad , \quad \forall\ s \in \mathbb{R}_+ \tag{5.3}$$

Denote such functions by $f(\cdot) \in PE^n(h,\beta)$.

It follows from the definition that if $\phi(\cdot) \in PE^p(h,\beta) \cap C_\delta^\tau$ then the rank condition (5.2) will hold for any $T \geq h \geq \delta$, and thus, (5.1) *may* be satisfied for some matrix P. The point to emphasize is that persistent excitation is not sufficient

for stability, except in the case when $H(s)$ is SPR [1]. Thus, we can view (5.1) as a *signal dependent positivity condition*. In general, the PE condition is necessary for stability, but as seen from (4.5) in Theorem 4.1, even if it holds, the system can be still be unstable.

(C) Parameter Scaling

The matrix P in (5.1) can be viewed as a *scaling* of the parameter vector. That is, if (5.1) holds for some P, then for all small $\varepsilon > 0$, $\dot\theta = -\varepsilon R(t)\theta$ is u.a.s. in the sense that $\theta(t)'P\theta(t) \to 0$ exponentially fast as $t \to \infty$. Thus, parameters will tend to converge with different scalings. If for some *given* signal $\phi(t)$, the determined scaling matrix P gives unwanted responses, then the signal can be reshaped so as to produce a more desirable scaling. The difficulty is in finding the matrix P. If $\phi(t)$ is almost periodic then Theorem 4.3 holds, and we can take P as the solution to $P\bar{A}+\bar{A}'P = -2I$. When $\phi(t)$ has a sample-average, there is no simple means to find P.

If there is sufficient a priori knowledge about the effect of parameters on the system, then this information will provide the *desired* scaling in the following sense. It is always possible to prescale θ and then select $P = \dfrac{1}{\alpha_0} I$ where α_0 is some positive constant. With this choice, condition (5.1) becomes,

$$\min_i \lambda_i \left\{ \sum_{\omega \in \Omega_+} H(-j\omega)\mathrm{Re}[X_k(\omega)] \right\} \geq \alpha_0 \quad , \quad \forall \; k \in \mathbb{Z}_+ \tag{5.4}$$

This is equivalently expressed as,

$$\min_i \lambda_i \left\{ \sum_{\omega \in \Omega_k} \mathrm{Re}[H(j\omega)]\mathrm{Re}[X_k(\omega)] \right\} \geq \alpha_0/2 \quad , \quad \forall \; k \in \mathbb{Z}_+ \tag{5.5}$$

which has a more informative interpretation in terms of the usual positivity conditions on H. For example, a strictly proper transfer function $H(s)$ is strictly positive real (SPR) if it is exponentially stable and $\exists$ constant $\rho > 0$ such that [16]:

$$\mathrm{Re}[\hat{H}(j\omega)] \geq \rho|\hat{H}(j\omega)|^2 \quad , \quad \forall \; \omega \in \mathbb{R}_+$$

This condition must hold at *every* frequency, whereas (5.5) requires $\mathrm{Re}[\hat{H}(j\omega)] > 0$ at those *discrete* frequencies in $\mathbb{R}_+$ where the magnitude of the input spectrum is large. Conversely, at those frequencies in $\mathbb{R}_+$ where $\mathrm{Re}[\hat{H}(j\omega)] < 0$, the magnitude of the input spectrum should be small. Since (5.5) will fail if $\mathrm{Re}\hat{H}(j\omega) < 0$, $\forall \; \omega \in \mathbb{R}_+$, it

follows that $\mathrm{Re}\hat{H}(j\omega) > 0$ at some frequencies, hence, the motivation to refer to (5.1) as a positivity condition.

(D) Bounds on ε

The upper bound ε_0 on the size of $\varepsilon > 0$ to insure stability can be extracted from the proof of Theorem 4.1. Looking back over Theorem 3.1, Theorem 2.1 and subsequent discussions we have,

$$\varepsilon_0 = \min\{\alpha/\beta, \varepsilon_1, \varepsilon_2\} \tag{5.6}$$

where ε_1 and ε_2 satisfy

$$\beta - \varepsilon_1 M\|\phi\|_\infty^4 K^2/(\alpha - \varepsilon_1\beta)^2 = 0 \tag{5.7}$$

$$\exp(-\varepsilon_2 T\alpha_0) + (\varepsilon_2 T\alpha_0)^2\exp(\varepsilon_2 T\alpha_0) = 1 \tag{5.8}$$

Recall from the proof of Theorem 3.1 that α_0, α, β, M, and K are defined from:

$$\mu\left[-\frac{1}{T}\int_{kT}^{(k+1)T} (\phi H\phi')(t)dt\right] \leq -\alpha_0 \quad, \quad \forall\ k \in \mathbb{Z}_+ \tag{5.9}$$

$$M = \exp(\varepsilon T(m + \beta)) \tag{5.10}$$

$$|h(t)| \leq K\exp(-\alpha t) \tag{5.11}$$

$$\exp(-\varepsilon T\beta) = \exp(-\varepsilon T\alpha_0) + (\varepsilon T\alpha_0)^2\exp(\varepsilon T\alpha_0) \tag{5.12}$$

(E) A Limitation Arising from Averaging

Suppose $H(s)$ is SPR and (4.4) holds. Hence, system (1.1) is exponentially stable for all *small* $\varepsilon > 0$. Since (4.4) holds, it follows that $\phi(t)$ is persistently exciting. However, from other arguments (see, e.g., [1]) we know that under these same conditions the zero solution of (3.1) is u.a.s. for *all* $\varepsilon > 0$. Thus, Theorem 4.1 is conservative in this case in regard to the limitations on ε. However, when $H(s)$ is not SPR Theorem 4.1 is now applicable whereas the results in [1] do not apply. In fact in this latter case when ε gets too large then system (1.1), can be unstable, even if (4.4) holds. For example, if in (1.1) $\phi(t) = \sin(0.35t)$ and $\hat{H}(s) = 1/(s^2+2s+2)$ then condition (4.4) is satisfied. The simulations in Fig 5.1 with $\theta(0) = 1$ show that the zero solution is u.a.s. for $\varepsilon = 4$ but is completely unstable for $\varepsilon = 8$.
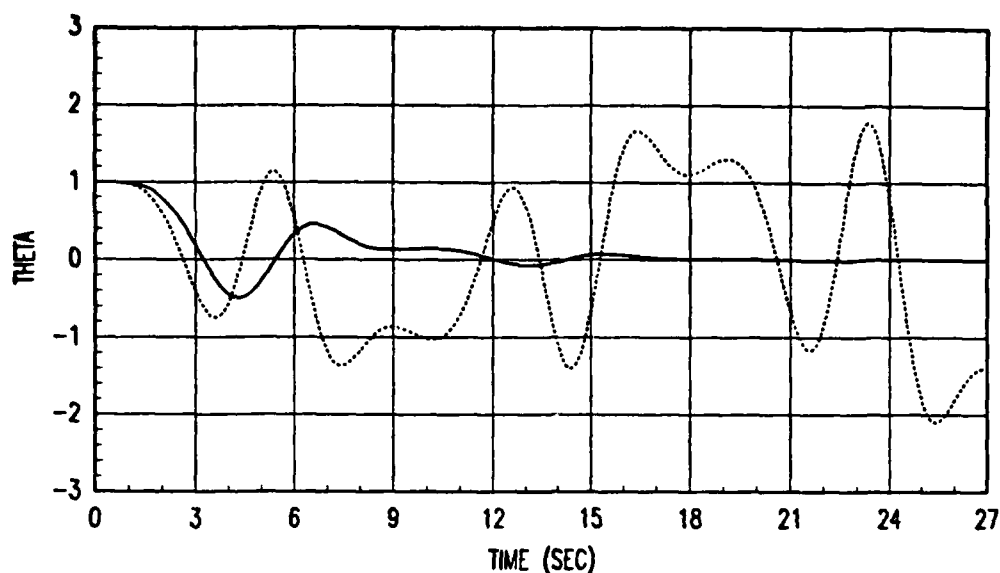
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Figure 5.1

Comparison with Averaging Analysis of Stochastic Recursive Algorithms

Comparing our results with the ordinary differential equation approach (ODE), used in the analysis of stochastic recursive algorithms, [22], [23] we notice the following differences:

(1) The ODE approach can deal with nonlinear recursions, whilst our analysis is restricted to the linear case. It is possible to extend our results to the nonlinear case, but this would introduce more technicalities (see e.g., [20]) perhaps obscuring the main idea of "local averages."

(2) In the ODE approach it is assumed that the adaptable gain (our ε) converges to zero (and is not summable), whilst in the present contribution ε is a small positive constant.

(3) The main difference lies in the condition imposed on the regressor vector sequence. Typically, the ODE approach relies on an ergodicity or mixing assumption to infer the existence of cesaro-mean along the sample paths ($\equiv$ average). Our conditions only involve finite sample path properties of the regressor vector and related

quantities; this makes the concept of cesaro-mean or global average meaningless. In this sense, the ODE approach is closer to [9a,b] where periodicity or almost periodicity are invoked to guarantee the existence of averages. In a way, our conditions allows for a second slow time scale, the slower time scale on which the nature of the regressor vector is allowed to change.

(4) The present approach yields instability results as well, a point not touched upon in the ODE approach.

6. CONCLUSION

We have presented a method of averaging for linear time varying systems, allowing one to deal with general time motions, thus removing the classical restriction of almost periodicity.

This method can be applied to the nonlinear adaptive control problem after linearizing the system in the neighborhood of the tuned solutions. Both (local) stability and instability have been discussed. The conditions obtained to guarantee local stability can be expressed in frequency domain terms.

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